ON THE VASSILIEV KNOT INVARIANTS

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ABSTRACT. The theory of knot invariants of finite type (Vassiliev invariants) is described. These invariants turn out to be at least as powerful as the Jones polynomial and its numerous generalizations coming from various quantum groups, and it is conjectured that these invariants are precisely as powerful as those polynomials. As invariants of finite type are much easier to define and manipulate than the quantum group invariants, it is likely that in attempting to classify knots, invariants of finite type will play a more fundamental role than the various knot polynomials.

CONTENTS

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1. Introduction

1.1. A sketchy introduction to the introduction. Any numerical knot invariant V can be inductively extended to be an invariant $V^{(m)}$ of immersed circles that have exactly m *transversal* self intersections using the formulas

$$V^{(0)} = V,$$

$$V^{(m)}\left(\times\right) = V^{(m-1)}\left(\times\right) - V^{(m-1)}\left(\times\right). \tag{1}$$

We know from multi-variable calculus that differences are cousins of derivatives, and so we wish to think of (1) as the definition of the mth partial derivative of a knot invariant in terms of its $(m-1)$st partial derivatives. (In a knot projection there can be many crossings, and so one can 'differentiate' with respect to many different 'variables'. Hence we think of (this corner of) knot theory as *multi*-variable calculus).

An invariant V is called *"(A Vassiliev invariant) of type m"* if its $(m+1)$st derivative vanishes identically[1]. Just like in calculus, $V^{(m+1)} \equiv 0$ implies that $V^{(m)}$ is a constant, in some sense. This 'constant' is actually a collection of constants corresponding to the different possible partial derivatives of V. From the special nature of the "function" V (it is a knot invariant), it follows that there are some relations among these constants. A collection of constants satisfying these relations is called a *weight system*. The purpose of this paper is to discuss the following questions:

- Which of the well-known knot invariants are Vassiliev invariants? (See theorems 2, 3, and 5).
- What are weight systems in a more precise language? What are the relations that a weight system has to satisfy? (See definitions 1.6, 1.7, 1.9, and theorem 6).
- Is it always possible to 'integrate' a weight system m times and get a knot invariant? (See theorem 1).
- Knot invariants can be added and multiplied. Knots can be multiplied (the operation of 'connected sum') and therefore knot invariants can be co-multiplied. What are the operations that one can perform on a weight system? Is the space of weight systems a Hopf algebra? (See theorems 7, 8, and 9).
- Can one classify all weight systems? (See theorems 4, 10, 11, 12 and conjectures 1 and 2).

Quite unexpectedly, given a representation of a semi-simple Lie algebra there is a simple construction of a weight system. In fact, a closer look at weight systems reveals that for many purposes they are as good as Lie algebras, despite their a-priori different appearance. The following further questions therefore arise:

- Which part of the theory of Lie algebras can be translated to the language of weight systems? (See *e.g.* section 5).
- Do *all* weight systems come from Lie algebras? (See the statistics in section 6.1, theorem 11 and conjectures 1 and 2).

The following outstanding problem *will not* be discussed in this paper:

Problem 1.1. Is there an analog of Taylor's theorem in our context — can an arbitrary knot invariant be approximated by Vassiliev invariants? Do Vassiliev invariants separate knots?

An affirmative answer to the above question will, of course, give tremendous further impetus to studying Vassiliev invariants.

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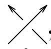
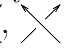
[1] Having the analogy with calculus in mind, a better name would have been "a polynomial invariant". But unfortunately, this name is already used for something else.

1.2. Acknowledgement. I wish to thank the many people who taught me about Vassiliev invariants, and the many people who were patient enough to listen to my own ideas about the subject. Among others, these include: V. I. Arnold, J. Birman, R. Bott, S. Duzhin, X-S. Lin, G. Masbaum, G. Wetzel, S. Willerton, and E. Witten. I am particularly grateful to M. Kontsevich, for inventing a considerable part of the theory described in these pages, for our many conversations, and for his careful reading of an earlier version of this paper.

1.3. Weight systems and invariants of finite type. Let $\mathbf{F}$ be a field of characteristic zero[2]. Any $\mathbf{F}$-valued invariant V of oriented knots in an oriented[3] three dimensional manifold M^3 can be extended to be an invariant of immersed circles in M^3, which are allowed to have some *transversal* self intersections, using the following definition:

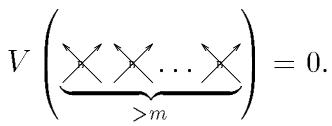
Definition 1.2. Suppressing the superfluous superscripts of (1), set

(2) 

As usual in knot theory and as will be the standard throughout the rest of this paper, when we write ,  or , we think of them as parts of bigger graphs which are identical outside of a small sphere, inside of which they look as in the figures. Notice also that in an oriented manifold the notion of an overcrossing (undercrossing) is well defined and does not depend on a choice of a planar projection. See for example [21, pp. 13].

Definition 1.3. Let m be a natural number. An invariant V of oriented knots in an oriented three dimensional manifold M^3 is called *an invariant of type m*, if V vanishes on "knots" that have more than m self intersections:

(the Birman-Lin condition)



An invariant V of oriented knots in an oriented three dimensional manifold M^3 is called a *Vassiliev* invariant [42, 43], or *an invariant of finite type*, if it is of type m for some $m \in \mathbf{N}$. The space $\mathcal{V}$ of all Vassiliev invariants is *filtered*, with $\mathcal{V}_m = \{\text{invariants of type } m\}$.

Remark 1.4. Vassiliev's original approach is very different from ours. He derives his class of invariants from certain topological considerations that involve viewing the space of all embeddings of S^1 into $\mathbf{R}^3$ as the complement of the space of singular

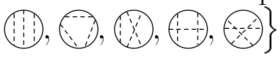
[2] In fact, a considerable fraction of the results proven here are true even with the field $\mathbf{F}$ replaced by an arbitrary Abelian group; see problem 7.3.

[3] Przytycki [34] noticed that Vassiliev invariants can be defined on non-orientable manifolds as well.

immersions in the space of all immersions. Our definition follows the 'Birman-Lin axioms' of [10] and can be extracted from Vassiliev's more involved theory.

Throughout the rest of this paper we will consider only the case of $M^3 = \mathbf{R}^3$. Let us now turn to the study of the mth 'derivative' of a Vassiliev invariant:

Definition 1.5. A *chord diagram* (CD) is an oriented circle with finitely many chords marked on it, regarded up to orientation preserving diffeomorphisms of the circle. Denote the collection of all chord diagrams by $\mathcal{D}^c$ (c for 'chord'). This collection is naturally graded by the number of chords in such a diagram. Denote the piece of degree m of $\mathcal{D}^c$ by $\mathcal{G}_m\mathcal{D}^c$. $\mathcal{G}_m\mathcal{D}^c$ is simply the collection of all chord diagram having precisely m chords.

By convention, we will always orient the circle in a chord diagram counterclockwise, and always use dashed lines for the chords. For examples, the collection $\mathcal{G}_3\mathcal{D}^c$ of chord diagrams of degree 3 is $\mathcal{G}_3\mathcal{D}^c = \Big\{$  $\Big\}$.

Definition 1.6. An $\mathbf{F}$-valued *weight system* of degree m is a function $W : \mathcal{G}_m\mathcal{D}^c \to \mathbf{F}$ having the following properties:

(1) If $D \in \mathcal{G}_m\mathcal{D}^c$ has an *isolated chord* — a chord that does not intersect any other chord in D, then $W(D) = 0$. This property is called *framing independence*.

(2) Whenever four diagrams S, E, W, and N differ only as shown in figure 1, their weights satisfy

$$W(S) - W(E) = -W(W) + W(N). \tag{3}$$

This property is called the $4T$ (4 Term) relation.

Let $\mathcal{W}$ denote the graded vector space of all weight systems.

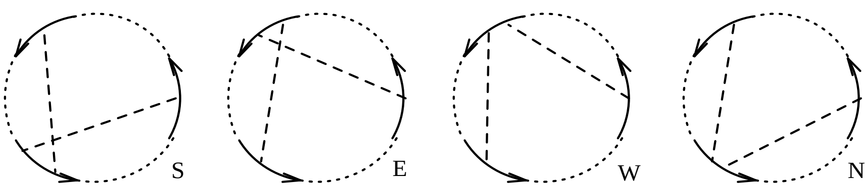


FIGURE 1. The diagrams S, E, W, and N. (The dotted arcs represent parts of the diagrams that are not shown in the figure. These parts are assumed to be the same in all four diagrams)

The first theorem of this paper says that over the real numbers the above two notions are essentially equivalent:

Theorem 1. *Over* $\mathbf{R}$, *the graded vector space associated with the filtered vector space* $\mathcal{V}$ *of Vassiliev invariants is* $\mathcal{W}$. *More precisely:*

(1) *(Proof on page 13) For a given non-negative integer m there is a naturally defined map $V \mapsto W_m(V)$ which to any given type m $\mathbf{F}$-valued Vassiliev invariant V associates a degree m $\mathbf{F}$-valued weight system $W_m(V)$.*

(2) *(Kontsevich [25]) (Proof on page 31) Restricting to the case $\mathbf{F} = \mathbf{R}$, there is a naturally defined map $W \mapsto V(W)$ which to any given $\mathbf{R}$-valued weight system of degree m associates a type m $\mathbf{R}$-valued Vassiliev invariant $V(W)$.*

(3) *(Proof on page 31) The above two maps are nearly each other's inverse — for any given W one has $W = W_m(V(W))$, and for any given V the invariants V and $V(W_m(V))$ differ by a knot invariant of type $m-1$.*

1.4. There are many invariants of finite type.

Theorem 2. *(Bar-Natan [5, 6]) (Proof on page 12). Each coefficient of the Conway polynomial is an invariant of finite type.*

Theorem 3. *(Birman-Lin, [10]) (Proof on page 12) After a suitable change of variables, each coefficient in the Taylor expansion of the Jones [20], HOMFLY[4] [19], and Kauffman [22] polynomials is an invariant of finite type.*

Theorem 4. *(Bar-Natan [5, 6]) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbf{F}$, t an Ad-invariant symmetric non-degenerate bilinear form on $\mathfrak{g}$, and R a finite dimensional representation of $\mathfrak{g}$. Let m be a non-negative integer.*

(1) *(Proof on page 17) Given this information, there is a natural construction of an associated functional[5] $W_{\mathfrak{g},R,m} : \mathcal{G}_m\mathcal{D}^c \to \mathbf{F}$ satisfying the $4T$ identity (3).*

(2) *(Proof on page 23) There is a canonical way to 'renormalize' $W_{\mathfrak{g},R,m}$ to a weight system $\hat{W}_{\mathfrak{g},R,m}$.*

Theorem 5. *(Lin, [28]) After a suitable change of variables, each coefficient in the Taylor expansion of the Reshetikhin-Turaev 'quantum-group' invariants [35, 36, 37, 41] corresponding to a quantization of a triple $\{\mathfrak{g}, t, R\}$ as in theorem 4 is an invariant of finite type. Furthermore (Piunikhin, [33]), the weight systems underlying those invariants are precisely those constructed in theorem 4. (See remarks 2.2, 2.3, and 4.8 and problem 4.9).*

[4]The HOMFLY polynomial is named after the initials of 6 of its discoverers, Hoste, Ocneanu, Millet, Freyd, Lickorish and Yetter. In fact, it was also discovered simultaneously by Przytycki and Traczwyk and therefore, following L. Rudulph, it should more accurately be called "the LYMPH-TOFU polynomial", with the last U standing for the Unknown further discoverers.

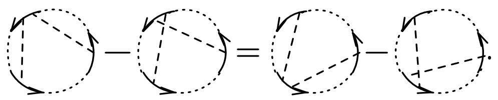
[5]t is normally taken to be given by matrix trace in the defining representation, and is always suppressed from the notation.

1.5. The algebra $\mathcal{A}$ of diagrams. For some purposes, it is better to consider weight systems as linear functionals over a graded vector space $\mathcal{A}^c$.

Definition 1.7. $\mathcal{A}^c$ is the quotient space

$$\mathcal{A}^c = \operatorname{span}(\mathcal{D}^c) \,/\, \operatorname{span}\{\text{all } 4T \text{ relations}\}\ .$$

Namely, $\mathcal{A}^c$ is the quotient of the vector space freely generated by all chord diagrams by the subspace spanned by all relations of the form:



As the $4T$ relation is homogeneous, $\mathcal{A}^c$ inherits a gradation from $\mathcal{D}^c$.

Clearly, a weight system of degree m is just a linear functional on $\mathcal{G}_m\mathcal{A}^c$ which vanishes on classes represented by a chord diagram that has an isolated chord.

The space $\mathcal{A}^c$ has an equivalent description as follows:

Definition 1.8. A *Chinese Character Diagram*[6] (CCD) is a connected graph made of a single oriented circle and a certain number of unoriented dashed lines, which are allowed to meet in two types of trivalent vertices:

(1) *Internal vertices* in which three dashed lines meet. These vertices are *oriented* — one of the two possible cyclic orderings of the arcs emanating from such a vertex is specified.
(2) *External vertices* in which a dashed line ends on the circle.

The collection of all Chinese character diagrams will be denoted by $\mathcal{D}^t$ (t for 'trivalent').

By convention, the circle in a CCD is always oriented counterclockwise, and so are the internal vertices. Also, as higher than trivalent vertices are not allowed in a CCD, what appears in a picture to be a vertex of order 4 is not a vertex at all — it is just a pair of arcs passing each other without intersection. An example is in figure 2.

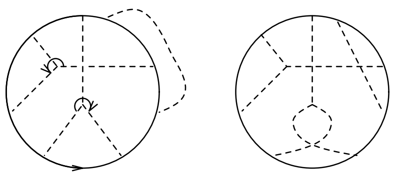
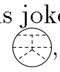
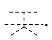


FIGURE 2. A Chinese Character Diagram (CCD) together with the conventional way of drawing it, as outlined in the text.

[6]This joke is due to Morton Brown. When the circle is stripped off a Chinese character diagram such as , what remains is a 'Chinese character' like .

Clearly, there is an even number of vertices in a CCD. We will use *half* the number of vertices to grade $\mathcal{D}^t$. A chord diagram is also a CCD, and it is easy to see that the gradations are compatible: $\mathcal{G}_m\mathcal{D}^c \subseteq \mathcal{G}_m\mathcal{D}^t$.

Definition 1.9. Let the vector space $\mathcal{A}^t$ be the quotient

$$\mathcal{A}^t = \operatorname{span}\left(\mathcal{D}^t\right)/\operatorname{span}\left\{\text{all } STU \text{ relations}\right\}.$$

An STU relation is a relation of the form $S = T - U$, where the diagrams S, T, and U are identical outside of a small circle, inside of which they look as in figure 3. As the STU relation is homogeneous, $\mathcal{A}^t$ inherits a gradation from $\mathcal{D}^t$.

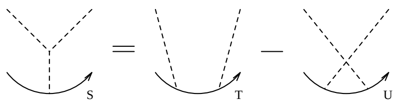
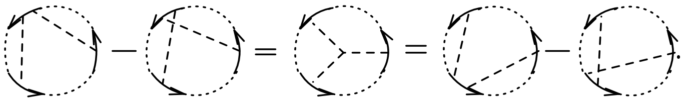


FIGURE 3. The diagrams S, T, and U, and the STU relation.

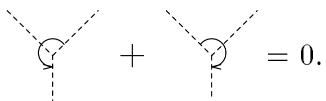
Remark 1.10. Notice that the $4T$ relation holds in $\mathcal{A}^t$:



(Both equalities in the above equation hold in $\mathcal{A}^t$ because of the STU relation). This implies that the inclusion $\mathcal{D}^c \hookrightarrow \mathcal{D}^t$ descends to a linear map $\phi^c : \mathcal{A}^c \to \mathcal{A}^t$.

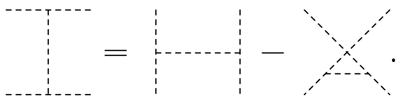
Theorem 6. *(Proof on page 17) The map $\phi^c : \mathcal{A}^c \to \mathcal{A}^t$ is an isomorphism. Furthermore, the following two identities hold in $\mathcal{A}^t$:*

(1) *Antisymmetry of internal vertices:*



(2) *The IHX identity:*

(4)



As $\mathcal{A}^t$ and $\mathcal{A}^c$ are anyway isomorphic, we will denote both (either) of them by the same symbol $\mathcal{A}$. Various pieces of the following theorem were discovered independently by Bar-Natan [5, 6], Kontsevich [25] and Lin [29]:

Theorem 7. *(Proof on page 18) $\mathcal{A}$ has a naturally defined product $\cdot$ and a naturally defined co-product Δ, which together make it a commutative and co-commutative Hopf algebra*[7]*. Furthermore, there is a naturally defined sequence of co-algebra automorphisms $\{\psi^q\}_{q=-\infty}^{\infty}$ of $\mathcal{A}$ reminiscent of the Adams operations of K-theory. These automorphisms satisfy $\psi^q \circ \psi^p = \psi^{qp}$.*

1.6. The primitive elements of $\mathcal{A}$. By the structure theory of Hopf algebras we know that $\mathcal{A}$ is the symmetric algebra generated by the primitive elements of $\mathcal{A}$:

$$\mathcal{A} = \mathcal{S}(\mathcal{P}(\mathcal{A})), \qquad \mathcal{P}(\mathcal{A}) = \{a \in \mathcal{A} : \Delta(a) = a \otimes 1 + 1 \otimes a\}. \tag{5}$$

The following description of $\mathcal{A}$ is better suited for the study of $\mathcal{P}(\mathcal{A})$:

Definition 1.11. A *Chinese Character* (CC) is a (possibly empty) graph whose vertices are either trivalent and oriented (in the sense of definition 1.8), or are univalent. The trivalent vertices in such a graph are called *internal*, while the univalent vertices are called *external*. The collection of all Chinese characters *that have at least one external vertex in each connected component* will be denoted by $\mathcal{C}$. It is graded by half the number of vertices in a character.

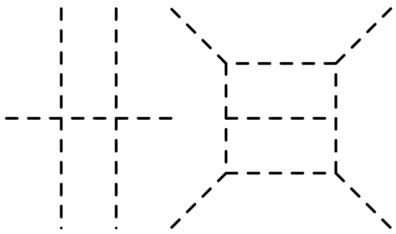


FIGURE 4. A Chinese Character (CC) of degree 8. As in the case of CCDs, all trivalent vertices are oriented counterclockwise and all apparent vertices of valency higher than 3 are not vertices at all. This CC has four connected components.

Let $\mathcal{B}$ be the quotient space

$$\mathcal{B} = \operatorname{span}(\mathcal{C}) / \{\text{anti-symmetric vertices and } IHX \text{ relations}\}. \tag{6}$$

Namely, $\mathcal{B}$ is just the vector space generated by Chinese characters with exactly the two relations of theorem 6 imposed. $\mathcal{B}$ inherits a grading from $\mathcal{C}$.

Theorem 8. *(Kontsevich [25]) (Proof on page 34) The spaces $\mathcal{A}$ and $\mathcal{B}$ are naturally isomorphic. Furthermore, if a space $\mathcal{P}$ is defined just like $\mathcal{B}$ only with connected Chinese characters replacing arbitrary Chinese characters, then the above isomorphism identifies $\mathcal{P}$ and $\mathcal{P}(\mathcal{A})$.*

[7]NOT in the graded (super) sense. Namely, elements in it honestly satisfy $a \cdot b = b \cdot a$, with no signs.

The dual space[8] $\mathcal{A}^*$ of the Hopf algebra $\mathcal{A}$ is also a Hopf algebra, and thus one might wish to investigate the primitive elements of $\mathcal{A}^*$. Let $\mathcal{P}'(\mathcal{A}^*)$ be the elements of degree different than 1 in $\mathcal{P}(\mathcal{A}^*)$.

Theorem 9. (1) *(Proof on page 23) $\mathcal{S}(\mathcal{P}'(\mathcal{A}^*))$ is isomorphic to the space $\mathcal{W}$ of all weight systems.*

(2) *(Lin [29], Kontsevich [25], Bar-Natan) (Proof on page 39) Via the correspondence of theorem 1, the weight systems in $\mathcal{P}'(\mathcal{A}^*)$ correspond to additive Vassiliev invariants — Vassiliev invariants which are additive under the operation of taking the connected sum of two knots.*

1.7. How big are $\mathcal{A}$, $\mathcal{W}$ and $\mathcal{P}$? One way to answer this question is by explicitly writing all diagrams and all relations, and using a computer to reduce the resulting matrix. The results are summarized in the table in page 39. Computer power is limited, however, and one might hope for better:

Conjecture 1. *All weight systems come from Lie algebras as in theorem 4. In other words, the weight systems produced in theorem 4 span the space of all weight systems.*

Definition 1.12. A *marked surface* is a compact two dimensional smooth surface with a choice of finitely many tangents (*markings*) to its boundary, regarded up to a diffeomorphism. (See figure 5). Let $\mathcal{M}$ be the vector space spanned by the set of marked surfaces that have at least one marking on each connected component.

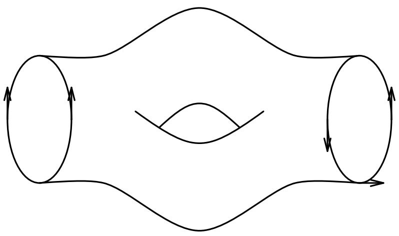


FIGURE 5. A marked surface with two boundary components

Theorem 10. *(Kontsevich [25], Bar-Natan) (Proof on page 42) There is a natural linear map $\Phi : \mathcal{B} \to \mathcal{M}$.*

Theorem 11. *(Proof on page 50) The pullback $(\Phi \circ \bar{\sigma})^* \mathcal{M}^*$ of $\mathcal{M}^*$ via $\Phi \circ \bar{\sigma}$ is the subring of $\mathcal{A}^*$ spanned by the linear functionals on $\mathcal{A}$ generated (as in theorem 4) by Lie groups in the families SO and GL and all of their representations. $(\Phi \circ \bar{\sigma})^* \mathcal{M}^*$ also contains in it the linear functionals on $\mathcal{A}$ generated by all representations of Abelian, symplectic and Spin groups.*

[8] $\mathcal{A}^*$ is the *graded dual* of $\mathcal{A}$ defined by $\mathcal{A}^* = \bigoplus (\mathcal{G}_m \mathcal{A})^*$.

Remark 1.13. The "philosophical father" of theorems 8, 10, and 11 is M. Kontsevich [25]. He suggested a somewhat weaker version of Φ, whose image is the space spanned by normalized orientable (in the sense of definition 6.11) marked surfaces, and conjectured theorem 11 for his version of Φ and only for the groups in the family GL. The stronger version of Φ, the fact that our Φ includes Kontsevich's (lemma 6.49) and the proofs of theorems 11 and 12 are due to the author.

Remark 1.14. There is a natural extension (See section 2.3) of the theory of Vassiliev invariants to framed links. Framed links have various natural *cabling* operations (see definitions 3.13 and 6.21 and exercise 6.43) and, dually, invariants of framed links have various natural cabling operations. These operations take Vassiliev invariants to Vassiliev invariants of the same type (see exercises 3.14, 6.22 and 6.43).

Theorem 12. *(Proof on page 56) The ring of Vassiliev framed knot invariants coming from the writhe, the HOMFLY and the Kauffman polynomials and all of their cablings corresponds via theorem* 1 *to the pullback* $(\Phi\circ\bar\sigma)^*\mathcal{M}^*$.

Conjecture 2. *(See the discussion in page 56) Linear functionals on $\mathcal{M}$ separate points in $\mathcal{A}$; that is, $\mathcal{A}^* = (\Phi\circ\bar\sigma)^*\mathcal{M}^*$. Theorem* 11 *implies that this conjecture is stronger than conjecture* 1.

1.8. Odds and ends. The last section of the paper contains an incomplete survey of the literature available on Vassiliev invariants, one piece of bad news — that conjecture 1 and the completeness of Vassiliev invariants (problem 1.1) cannot be true at the same time, and some questions.

1.9. Summary of spaces and maps.

1.9.1. *Spaces.*

$$\overbrace{\hspace{14em}}^{\text{collectively called } \mathcal{A} \text{ (theorem 6)}}$$

$$\begin{tikzcd}[column sep=large]
\substack{\text{chord diags.}\\ \text{def. 1.7}} & \substack{\text{trivalent diags.}\\ \text{def. 1.9}} & \substack{\text{linear diags.}\\ \text{eq. (8)}} & \substack{\text{Chinese chars.}\\ \text{eq. (6)}} & \substack{\text{marked diags.}\\ \text{def. 6.3}} & \substack{\text{marked surfs.}\\ \text{def. 1.12}} \\
\mathcal{A}^c \arrow[r, shift left, "\phi^c:\text{inclusion}"] & \mathcal{A}^t \arrow[l, shift left, "\text{theorem 3.1}"] \arrow[r, shift left, "\text{lemma 3.1}"'] & \mathcal{A}^l \arrow[l, shift left, "\phi^l:\substack{\text{closing to}\\ \text{a circle}}"'] \arrow[r, shift left, "\bar\sigma:\text{theorem 8}"'] \arrow[dd, "\mathcal{T}_\mathcal{A}"'] & \mathcal{B} \arrow[l, shift left, "\bar\chi:\substack{\text{sum over}\\ \text{orderings}}"'] \arrow[r, "\mu:\text{marking}", "\text{1-1?}"'] \arrow[dd, "\mathcal{T}_\mathcal{B}"] \arrow[rr, bend right=40, "\Phi", "\text{conjecture 2: 1-1?}"'] & \mathcal{B}^m \arrow[r, "\tau:\text{thickening}", "\substack{\text{isomorphism by}\\ \text{theorem 14}}"'] & \mathcal{M} \arrow[d] \\
 & & & \arrow[l, "\substack{\text{given}\\ \mathfrak{g},\, t\\ \text{(ex. 5.1)}}"'] \arrow[r] & \text{sect. 6.4} & \langle\cdot,\cdot\rangle \\
 & \mathbf{F} \arrow[from=uu, "W_{\mathfrak{g},R,m}"] & \mathfrak{U}(\mathfrak{g})^\mathfrak{g} \arrow[l, "\text{given } R"'] \arrow[r, equal, "PBW"'] & \mathfrak{S}(\mathfrak{g})^\mathfrak{g} & & \mathcal{L} \arrow[u]
\end{tikzcd}$$

1.9.2. *Dual spaces.*

$$\begin{tikzcd}[column sep=huge]
\substack{\text{Vassiliev invts.}\\ \text{def. 1.3}} & \substack{\text{weight systems}\\ \text{def. 1.6}} & & \\
\mathcal{V} \arrow[r, shift left, "\text{theorem 1}"] & \mathcal{W} \arrow[l, shift left, "\substack{W_m:\ \text{``differentiation'' defined up to type } m\\ V:\ \text{``integration''}}"] & \mathcal{A}^* \arrow[l, "\text{thm. 9 part 1:}", "\substack{\text{no real loss}\\ \text{of information}}"'] & \mathcal{M}^* \arrow[l, "(\Phi\circ\bar\sigma)^*", "\substack{\text{conjecture 2: onto?}\\ \text{image analyzed in thm. 11, 12}}"']
\end{tikzcd}$$

2. The basic constructions

2.1. The classical knot polynomials.

2.1.1. *Proof of theorem* 2. Let $C(K)(z)$ be the Conway polynomial of a knot K, and use the same symbol C to denote the natural extension (2) of the Conway polynomial to knots having self intersections. Then by the definition of the Conway polynomial [13, 21],

$$C\left(\times\right) = C\left(\nwarrow\!\!\!\!\nearrow_{+}\right) - C\left(\nwarrow\!\!\!\!\nearrow_{-}\right) = z \cdot C\left(\big)\big(\right).$$

Therefore if K has more than m double points, $C(K)$ is divisible by at least z^{m+1} and hence the coefficient of z^m in $C(K)$ vanishes. This implies that the mth coefficient of the Conway polynomial is a Vassiliev invariant of type m. $\square$

2.1.2. *Proof of theorem* 3. The idea is exactly the same as in the previous proof. In one of its standard parametrizations, the HOMFLY polynomial [19] is a function P of two parameters q and N which satisfies the identity

$$q^{N/2} P\left(\nwarrow\!\!\!\!\nearrow_{+}\right) - q^{-N/2} P\left(\nwarrow\!\!\!\!\nearrow_{-}\right) = (q^{1/2} - q^{-1/2}) P\left(\big)\big(\right).$$

Clearly if one changes variables to $q = e^x$ and expands in powers of x, the above equation can be rewritten in the form

$$P\left(\nwarrow\!\!\!\!\nearrow_{+}\right) - P\left(\nwarrow\!\!\!\!\nearrow_{-}\right) = x \cdot (\text{some mess}).$$

The precise form of the 'mess' in the right side of the above equation is immaterial. What ever it is, the same argument as in the previous proof carries through and we see that (in this parametrization) the mth coefficient of the HOMFLY polynomial is a Vassiliev invariant of type m. The Jones polynomial is just the $N \equiv 2$ specialization of the HOMFLY polynomial, and thus the same proof works for the Jones polynomial as well. Similar considerations also work in the case of the Kauffman polynomial. $\square$

Remark 2.1. The Conway polynomial proof and idea to substitute $q = e^x$ and expand in powers on x are due to the author. The argument in the HOMFLY case was completed by Birman and Lin in [10].

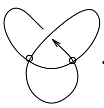
Remark 2.2. Similar arguments can be used to prove theorem 5. A complete proof can be found in Lin, [28] and Piunikhin, [33]. See also remark 4.8.

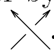
Remark 2.3. In fact, similar arguments can prove a slightly stronger theorem — that every knot invariant coming from a deformation of the identity solution of the Yang-Baxter equation can be re-expressed in terms of Vassiliev invariants.

2.2. Constructing a weight system from a Vassiliev knot invariant. Let V be a knot invariant of type m, and let $D \in \mathcal{G}_m\mathcal{D}^c$ be a chord diagram of degree m. An *embedding* of D in $\mathbf{R}^3$ will be an immersion $K_D : S^1 \to \mathbf{R}^3$ of the circle into $\mathbf{R}^3$ whose only singularities are transversal self intersections and which satisfies:

$$K_D(\theta) = K_D(\theta') \quad \Longleftrightarrow \quad (\theta = \theta') \text{ or } \begin{pmatrix} \theta \text{ and } \theta' \text{ are the two} \\ \text{ends of a chord in } D \end{pmatrix}.$$

For example,

$D =$  $\longrightarrow$ $K_D =$  .

If K_D and $\bar{K}_D$ are two embeddings of D, then one can get from one to the other by a sequence of 'flips', in which an overcrossing  is replaced by an undercrossing . Each such flip *does not* change the value of $V(K_D)$ — by definition 1.2 the change in the value of $V(K_D)$ when such a flip is performed is given by V evaluated on a knot with $m+1$ double points, and this is 0 by the Birman-Lin condition of definition 1.3. Thus one can unambiguously set

$$W(D) = W_m(V)(D) = V(K_D).$$

Proof of part (1) of theorem 1. We need to show that W is a weight system. First consider the case where D has an isolated chord. Then K_D can be chosen to look like in figure 6. Using equation (2) on the point P of figure 6, we get

$$W(D) = V(K_D) = V(K_D^o) - V(K_D^u),$$

where K_D^o (K_D^u) is the version of K_D in which the double point P was replaced by an overcrossing (undercrossing). But K_D^o and K_D^u are ambient isotopic, and therefore $W(D) = 0$.

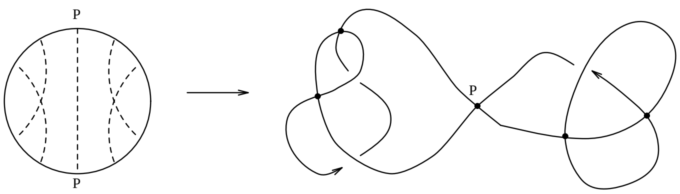


FIGURE 6. The knot corresponding to a diagram having an isolated chord. The ends of the isolated chord are mapped to the point P.

Next let SW be the almost saturated (*i.e.* having $m-1$ self-intersections) knot shown (partially) in figure 7. Pieces of the x and y axes near the origin serve as arcs in that knot, as well as a third line z' parallel to the z axis but transversing the $x-y$ plane South-West of the origin. Let NE be the same, only with the third line z' moved to transverse the $x-y$ plane North-East of the origin. There are two ways to calculate $V_i(NE)$ in terms of $V_i(SW)$ and the weights of saturated knots (knots having precisely m double points) using the flip relation — by moving z' from SW to NE along the two dotted paths in figure 7. The two ways must yield the same answer, and therefore the four saturated knots corresponding to z' intersecting the x and y axes South, East, West and North of the origin have diagrams whose weights are related. With the sign convention of (2), this relation is seen to be (3). $\square$

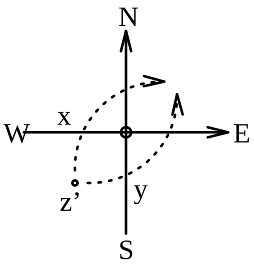


FIGURE 7. The graph SW and the two ways of getting from it to NE. Notice that z' is perpendicular to the plane and therefore appears as a single dot.

2.3. Framed links.

Definition 2.4. A framed knot is a knot K together with a choice of homotopy class in the space of never vanishing sections of the normal bundle of K. A framed link is a set of disjoint framed knots.

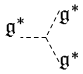
Exercise 2.5. Check that there is no difficulty in extending definitions 1.2 and 1.3 to framed knots. Show that the construction in the previous section carries through with only one change — if V is a Vassiliev invariant of framed knots and D has an isolated chord, $W_m(V)(D)$ does not necessarily vanish.

Exercise 2.6. Show that there is a simple extension of definitions 1.2 and 1.3 and of the theory of the previous section to framed links. The main difference is that chord diagrams will now have many circles instead of just one.

2.4. Constructing invariant tensors from Lie algebraic information. In this section we will discuss a general method for constructing invariant tensors from a certain type of diagrams and some Lie algebraic information. When restricted to chord diagrams, this construction produces weight systems.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbf{F}$, t an Ad-invariant symmetric non-degenerate bilinear form (a *metric*) on $\mathfrak{g}$ (the Killing form of $\mathfrak{g}$ is an example if $\mathfrak{g}$ is semi-simple), and let R a finite dimensional representation of $\mathfrak{g}$. The objects $(\mathfrak{g}, t, R)$ can all be regarded as tensors:

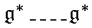
The Lie algebra. A Lie algebra $\mathfrak{g}$ is a vector space (also denoted by $\mathfrak{g}$) together with a distinguished element (tensor) $\tilde{f}$ of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ called *the Lie bracket*, subject to some well known requirements called *antisymmetry* and the *Jacobi identity*. The bilinear form t induces an isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$, and so $\tilde{f}$ corresponds to some tensor $f \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*$. The tensor f is totally antisymmetric. We will represent it by a graph:



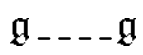
$$\longleftrightarrow \qquad f \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^*.$$

(The internal vertex in the above graph should be oriented, otherwise it is not clear which order its external vertices should be taken with so that they correspond to the three $\mathfrak{g}^*$'s on the r.h.s. of the correspondence. As usual, if no orientation is specified, we pick the counterclockwise orientation.)

The bilinear form. The bilinear form t is a tensor in $\mathfrak{g}^* \otimes \mathfrak{g}^*$. Its inverse is a tensor $t^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$. We will represent these two tensors by graphs:

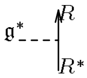


$$\leftrightarrow \quad t \in \mathfrak{g}^* \otimes \mathfrak{g}^* \quad ;$$



$$\leftrightarrow \quad t^{-1} \in \mathfrak{g} \otimes \mathfrak{g}.$$

The representation. A representation R is a vector space (also denoted by R) together with a distinguished tensor r in $\mathfrak{g}^* \otimes R \otimes R^*$. We will represent it by a graph:



$$\longleftrightarrow \qquad r \in \mathfrak{g}^* \otimes R \otimes R^*.$$

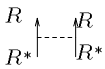
For completeness, we will also have a graphical representation for the identity $I \in \operatorname{End}(R)$:

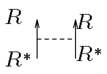
$$\longleftrightarrow \qquad I \in R^* \otimes R.$$

Let D be a diagram made of those components as above — dashed lines, directed full lines, oriented trivalent vertices in which three dashed lines meet, trivalent vertices in which a dashed line ends on a directed full line, 'beginnings' of full lines marked by an R^*, 'ends' of full lines marked by an R, and 'ends' of dashed lines marked by either a $\mathfrak{g}$ or a $\mathfrak{g}^*$. To such a diagram we associate a tensor

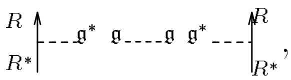
$$\mathcal{T}(D) = \mathcal{T}_{\mathfrak{g},R}(D) \in \bigotimes_{\substack{\text{univalent vertices} \\ v \text{ of } D}} \begin{pmatrix} \text{the vector space} \\ \text{marked near } v \end{pmatrix}.$$

The construction of $\mathcal{T}(D)$ is simple. Simply separate D to a 'union' of its components, consider what you've got as a tensor in some higher tensor product of the spaces involved, and contract the obvious pairs of spaces and their duals.

Example 2.7. We get a tensor $\Omega =$  $\in R \otimes R^* \otimes R \otimes R^*$ via

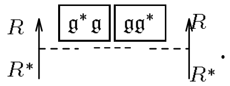


$$\longrightarrow$$



,

surrounding by a box pairs of spaces that are to be contracted, we get

$\longrightarrow$ 

In other words, Ω is the image of $r \otimes t^{-1} \otimes r \in (\mathfrak{g}^* \otimes R \otimes R^*) \otimes (\mathfrak{g} \otimes \mathfrak{g}) \otimes (\mathfrak{g}^* \otimes R \otimes R^*)$ under the map

$$(\mathfrak{g}^* \otimes R \otimes R^*) \otimes (\mathfrak{g} \otimes \mathfrak{g}) \otimes (\mathfrak{g}^* \otimes R \otimes R^*) \xrightarrow[\text{and spaces \#5 and \#6}]{\text{contract spaces \#1 and \#4}} R \otimes R^* \otimes R \otimes R^*.$$

Example 2.8. If we start from a *closed* diagram — a diagram with no external vertices, we just get a scalar in the ground field $\mathbf{F}$. It is an instructive exercise to verify that if $\{\mathfrak{g}_a\}_{a=1}^{\dim \mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ relative to the scalar product t, then

$$\mathcal{T}(\bigcirc) = \sum_{a,b,c=1}^{\dim \mathfrak{g}} f_{abc} \operatorname{tr}\left(R(\mathfrak{g}_a) R(\mathfrak{g}_b) R(\mathfrak{g}_c)\right) \in \mathbf{F},$$

where f_{abc} are the structure constants of $\mathfrak{g}$ with respect to the basis $\{\mathfrak{g}_a\}$.

Example 2.9. $\mathcal{T}\left(\bigcirc\right) = \operatorname{tr} I = \dim R$.

The Lie algebra $\mathfrak{g}$ acts on the spaces $\mathfrak{g}$, $\mathfrak{g}^*$, R, R^*, and therefore also on their various tensor products. As f, t^{-1}, and r are invariant tensors and the contraction operation is invariant, we see that:

Proposition 2.10. *The tensors $\mathcal{T}(D)$ are invariant under the above mentioned $\mathfrak{g}$-action.*

The tensors $\mathcal{T}(D)$ are not necessarily all different. Let $\mathcal{D}^a$ (a for 'all') denote the collection of all diagrams made of the above mentioned components, and let $\mathcal{A}^a$ be the vector space

$$\mathcal{A}^a = \operatorname{span}(\mathcal{D}^a) / \{\text{anti-symmetric vertices, } STU \text{ and } IHX \text{ relations}\}.$$

Proposition 2.11. *The map $\mathcal{T} : \mathcal{D}^a \to \{\text{invariant tensors}\}$ defined by $D \mapsto \mathcal{T}(D)$ descends to a map $\mathcal{T} : \mathcal{A}^a \to \{\text{invariant tensors}\}$.*

Proof. The Jacobi identity is the fact that a certain sum of three quadratic expressions in the structure constants of $\mathfrak{g}$ vanishes. Each of the diagrams making the IHX relation (4) is such a quadratic expression, and it is easy to check that the IHX relation (in the context of tensors produced using a Lie algebra) is just a restatement of the Jacobi identity. Similarly, it can easily be seen that the STU relation is the fact that 'representations represent'. Namely, it is just a restatement of the identity $R([a, b]) = R(a)R(b) - R(b)R(a)$ for $a, b \in \mathfrak{g}$. Antisymmetry of internal vertices is the total antisymmetry of the tensor f. $\square$

Proof of part (1) of theorem 4. Notice that a diagram $D \in \mathcal{D}^t$ is closed, and so $\mathcal{T}(D)$ is a scalar. Let $W : \mathcal{D}^t \to \mathbf{F}$ denote the map $D \mapsto \mathcal{T}(D)$. We have just proven that W descends to a linear functional $W \in (\mathcal{A}^t)^*$. Using remark 1.10 we see that we can restrict this linear functional to $\mathcal{G}_m\mathcal{A}^c$. Call the resulting functional $W_{\mathfrak{g},R,m}$. $\square$

Remark 2.12. The discussion in this section has an obvious generalization to the case where each chain or cycle of directed arcs is associated with (is *colored* by) a (possibly) different representation of $\mathfrak{g}$. Furthermore, using linear extension cycles of directed arcs can be colored by an arbitrary *virtual* representation (and the resulting extension is consistent in the obvious sense). For what might be an even further generalization, see problem 5.4.

3. The algebra $\mathcal{A}$ of diagrams

In this section we will prove the theorems of section 1.5.

3.1. Proof of theorem 6. Let us start by proving that the linear map $\phi^c : \mathcal{A}^c \to \mathcal{A}^t$ is an isomorphism. We will do that by constructing an inverse $\phi^t : \mathcal{A}^t \to \mathcal{A}^c$ — a map $\tilde{\phi}^t : \mathcal{D}^t \to \mathcal{A}^c$ satisfying the STU relation and extending the natural projection $\mathcal{D}^c \to \mathcal{A}^c$. To do that, notice that the STU relation expresses a diagram with some number k of internal vertices as a difference of diagrams with just $k-1$ internal vertices. Using the STU relation repeatedly it is clear how to construct $\tilde{\phi}^t$ inductively, and the only problem is to show consistency — that if a CCD is reduced via STU to a linear combination of chord diagrams in two different ways, the resulting combinations are equivalent mod $4T$.

If $D \in \mathcal{D}^t$ has only one internal vertex, consistency is clear — it is precisely the $4T$ relation. So let us assume that $\tilde{\phi}^t$ has been successfully defined on diagrams with less than k internal vertices, $k > 1$, and let D be a diagram with exactly k internal vertices. Suppose the STU relation is used to express D as a difference of two diagrams with $k-1$ internal vertices in two ways — by applying it to remove the arcs i or j, each of which connects the circle to an internal vertex. If i and j are not connected to the same internal vertex figure 8 shows why the two ways agree.

If i and j are connected to the same internal vertex, pick a third arc l that connects the circle to a different internal vertex (if possible), and use the transitivity of the equality relation. There is one exceptional case[9] in which such a third arc l does not exist. It is somewhat more convenient to return to this case later, after the proof of lemma 3.1, on page 20.

It remains to show that the STU relation implies the antisymmetry of internal vertices and the IHX identity. For the latter, by repeatedly using STU (if necessary), we can assume that the figures I, H, and X that we are dealing with touch the circle in one of their corners, say the lower left corner. The proof is now similar to the

[9]Of which I was informed by G. Masbaum after a preliminary version of this paper was circulated.

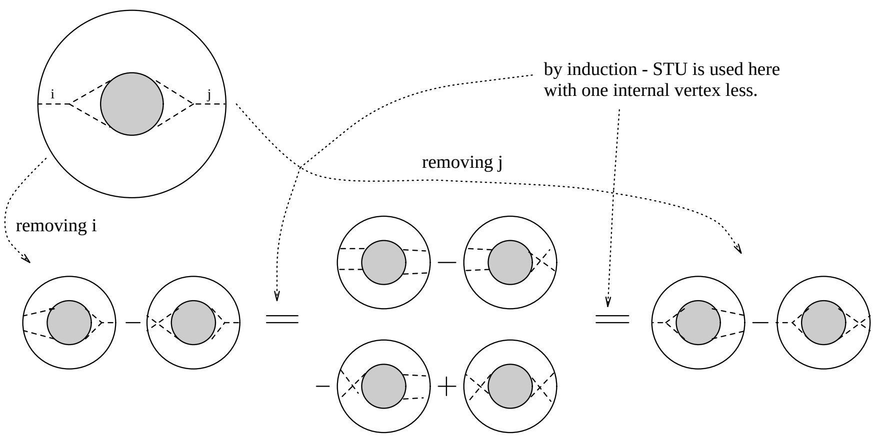


FIGURE 8. The consistency proof

proof that the commutator in an associative algebra satisfies the Jacobi identity, and is summarized in figure 9.

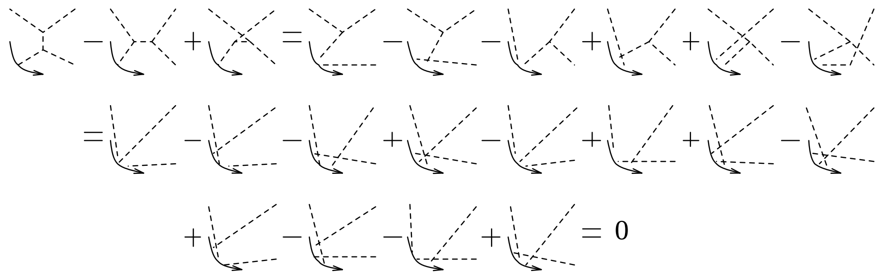
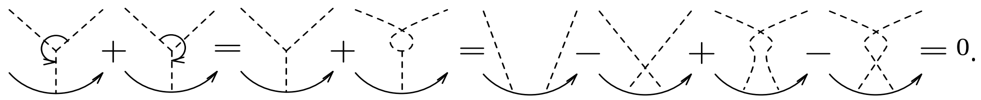


FIGURE 9. IHX follows from STU

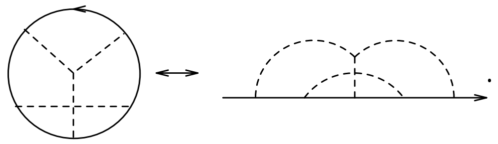
Similarly, the antisymmetry of internal vertices follows from:



□

3.2. Proof of theorem 7. For the purpose of this proof, it is convenient to 'linearize' diagrams:

(7)



More precisely, define $\mathcal{D}^l$ to be the collection of all 'linear' diagrams — these are the same as Chinese character diagrams only with the circle replaced by a directed line.

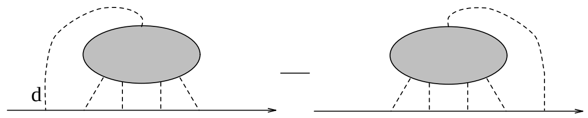
Then define $\mathcal{A}^l$ to be

$$\operatorname{span}\left(\mathcal{D}^l\right) / \operatorname{span}\{\text{all } STU \text{ relations}\}\ . \tag{8}$$

Clearly there is a map $\phi^l : \mathcal{A}^l \to \mathcal{A}$ defined by 'closing the line into a circle' as in (7).

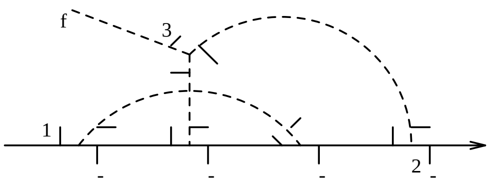
Lemma 3.1. *The map ϕ^l is an isomorphism.*

Proof. The surjectivity of ϕ^l is trivial, and all that is required is to prove its injectivity — that two 'linear' diagrams that map to the same 'circular' diagram are in fact equal as members of $\mathcal{A}^l$:

(9) 

$\stackrel{?}{=} 0.$

This is easily accomplished. Disconnect the vertex marked by the letter d in (9). Add three little 'right turning hooks' near each of the remaining vertices. Then put a $(-)$ sign near each of the hooks that is connected to a directed arc *leaving* the vertex. What you get looks like this:

(10) 

Consider the signed sum Σ of all possible ways of connecting the 'floating end' (marked by the letter f) to one of the 15 (in this case) hooks, taking hooks marked with a $(-)$ with a negative sign. There are two ways to group the 15 terms in this sum, and comparing these two ways will prove our lemma.

(1) By arcs: our sign convention and the antisymmetry of internal vertices show that when the terms in Σ are grouped by arcs, all drops out (see figure 10) except for the terms corresponding to the three 'groupless' hooks. These are marked by 1, 2, and 3 in (10). The first two of these three terms form exactly the left hand side of (9), while the third vanishes because of the antisymmetry of internal vertices.

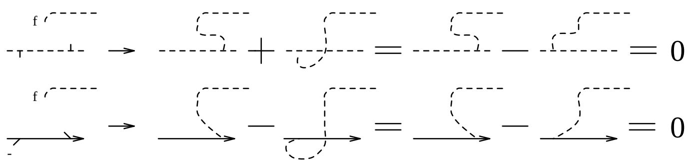


FIGURE 10. Grouping by arcs

(2) By vertices: our sign convention, the antisymmetry of internal vertices, and the STU and IHX relations show that when the terms in Σ are grouped by vertices, all drops out (see figure 11). □

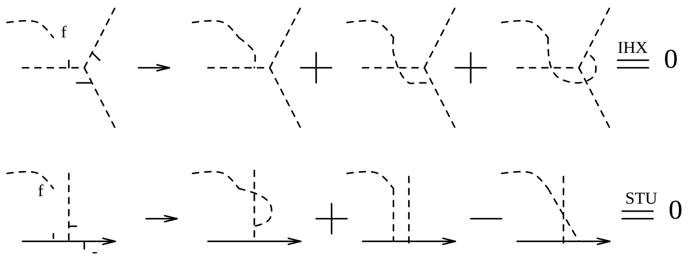
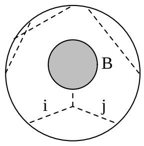


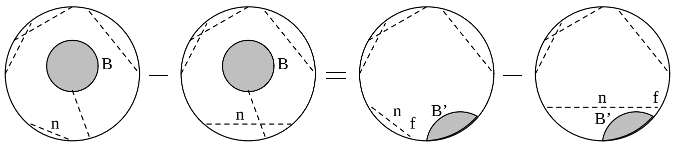
FIGURE 11. Grouping by vertices

Remark 3.2. From the knot theory view point, lemma 3.1 is just the fact that the theory of knots with a distinguished point is equivalent to the standard theory of knots. In [10], Birman and Lin have used this view point to prove a special case of lemma 3.1.

Conclusion of the proof of theorem 6: The one exceptional case alluded to in the proof of theorem 6 is when the arcs i and j are connected to the same internal vertex and there are no other arcs connecting an internal vertex to the circle. In that case, D must look as follows:

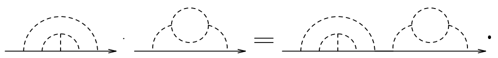
$D =$  (no trivalent vertices other than inside the blob B, and the one connecting i and j.) .

Fortunately, no matter how STU is used to reduce D to chord diagrams, the answer is 0 mod $4T$. Indeed, applying STU around (say) i and then collapsing B into chord diagrams using STU relations, we get

$D =$  $= D_T - D_U.$

But D_T and D_U are equal — to get from D_T to D_U one has to pass the end f of the chord marked n over the chords in the blob B'. This is possible (modulo the $4T$ relation) by an argument parallel to the argument in the proof of lemma 3.1. The only differences are that hooks are put only on the circle, and that instead of summing STU and IHX relations over vertices, one sums $4T$ relations over dashed arcs. □

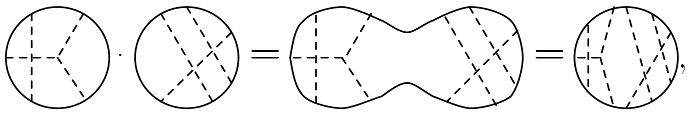
Definition 3.3. The product $\cdot : \mathcal{D}^l \otimes \mathcal{D}^l \to \mathcal{D}^l$ is the operation of *connected sum*,



Proposition 3.4. *The above defined product $\cdot$ descends to a product $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. With this product, $\mathcal{A}$ is an associative and commutative algebra. The empty circle $\bigcirc$ is a unit in $\mathcal{A}$.*

Proof. Remembering that $\mathcal{A} \simeq \mathcal{A}^l$, the only problem that remains is to show the commutativity of the product as an operation $\mathcal{A}^l \otimes \mathcal{A}^l \to \mathcal{A}^l$. But clearly for $D_{1,2} \in D^l$, $D_1 \cdot D_2$ and $D_2 \cdot D_1$ correspond to the same diagram in $\mathcal{D}^t$ under the isomorphism ϕ^l. $\square$

Remark 3.5. As an operation on $\mathcal{D}^t$, the product is also given as connected sum,



It is *not* well defined as an operation $\mathcal{D}^t \otimes \mathcal{D}^t \to \mathcal{D}^t$. As such, it depends on the points on circles used for the surgery involved in the connected sum. However, as an operation $\mathcal{D}^t \otimes \mathcal{D}^t \to \mathcal{A}$ it is well defined, as follows from the equivalence of $\mathcal{A}^l$ and $\mathcal{A}^t$.

Exercise 3.6. Check that the product of $\mathcal{A}$ corresponds in a natural way to the *connected sum* operation on knots.

Definition 3.7. Let $\mathcal{D}$ be the vector space generated by the elements of $\mathcal{D}^t$. Define a *splitting* D_s of a diagram $D \in \mathcal{D}^t$ to be a marking of the dashed arcs in D by the letters l and r, so that if three dashed arcs of D meet at some vertex, then they are all marked by the same letter. Define $\Delta : \mathcal{D}^t \to \mathcal{D} \otimes \mathcal{D}$ by

$$\Delta(D) = \sum_{\substack{\text{all splittings} \\ D_s \text{ of } D}} L(D_s) \otimes R(D_s),$$

where $L(D_s)$ ($R(D_s)$) is obtained from D by removing all the arcs marked by r (l) in D_s. See figure 12.

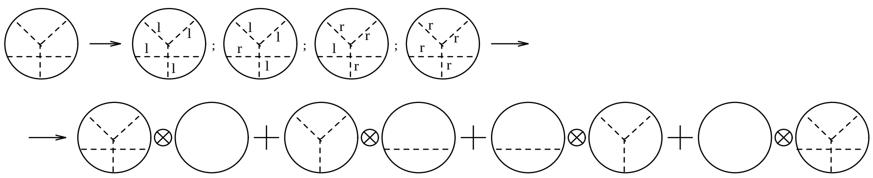


FIGURE 12. Computing $\Delta\left(\bigcirc\right)$.

Proposition 3.8. Δ *descends to a co-commutative co-associative co-product on* $\mathcal{A}$. *The linear functional* $\epsilon \in \mathcal{A}^*$ *defined by* $\epsilon\left(\bigcirc\right) = 1$; $\epsilon|_{\mathcal{G}_m\mathcal{A}} = 0$, *($m > 0$) is a co-unit in the resulting co-algebra.*

Proof. An explicit computation shows that

This implies that Δ is well defined as a map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. The other assertions of the proposition are trivial. □

Proposition 3.9. *With the operations defined above,* $\mathcal{A}$ *is a Hopf algebra.*

Proof. Let $D_{1,2} \in \mathcal{D}^t$. The compatibility $\Delta(D_1D_2) = \Delta(D_1)\Delta(D_2)$ of the product and the co-product is just the statement

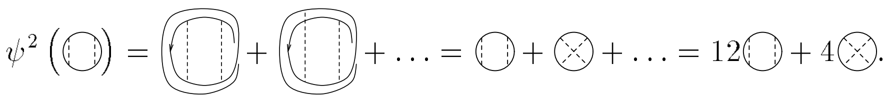
$$\sum_{\substack{\text{all splittings}\\(D_1D_2)_s \text{ of } D_1D_2}} L((D_1D_2)_s) \otimes R((D_1D_2)_s) = \sum_{\substack{\text{all splittings}\\ D_{1s} \text{ of } D_1\\ D_{2s} \text{ of } D_2}} L(D_{1s})L(D_{2s}) \otimes R(D_{1s})R(D_{2s}),$$

which is rather clear. □

Exercise 3.10. Show that the co-product of $\mathcal{A}$ corresponds to the multiplication of knot invariants. In other words, let V_1 (V_2) be a Vassiliev invariant of type m_1 (m_2). Show that $V_1 \cdot V_2$ is a Vassiliev invariant of type $m_1 + m_2$ and that

$$W_{m_1+m_2}(V_1 \cdot V_2) = (W_{m_1}(V_1) \otimes W_{m_2}(V_2)) \circ \Delta.$$

Definition 3.11. Let $D \in \mathcal{D}^t$ be a diagram, and let $q \in \mathbf{Z}$ be a positive integer. Define $\psi^q(D) \in \mathcal{D}$ to be the sum of all possible ways of lifting D to the qth cover of the circle. For example,



For $q = 0$, just define

$$\psi^0(D) = \begin{cases} \bigcirc & \text{if } D = \bigcirc, \\ 0 & \text{otherwise,} \end{cases}$$

and for negative q define $\psi^q(D) = (-1)^n \psi^{-q}(D)$, where n is the number of vertices on the circle of D.

Exercise 3.12. Show that ψ^q descends to a co-algebra automorphism of $\mathcal{A}$ and that for any $q, p \in \mathbf{Z}$, $\psi^q \circ \psi^p = \psi^{qp}$.

Notice that propositions 3.4, 3.8, 3.9 and exercise 3.12 prove theorem 7.

Definition 3.13. Let $K : (S^1 = \{z : |z| = 1\}) \to \mathbf{R}^3$ be a framed knot and let $n(z)$ $(z \in S^1)$ be a section of the normal bundle of K compatible with its framing. Notice that the normal bundle of K has a natural complex structure. For a non-zero integer q define the qth *connected cabling* $\psi^q(K)$ of K to be the knot given by

$$\psi^q(K)(z) = K(z^q) + \epsilon z n(z^q),$$

where ϵ is a very small number. Let $\psi^q(K)$ inherit the framing of K.

Exercise 3.14. Show that the operations ψ^q on knots and on diagrams correspond. In other words, show that if V is a Vassiliev invariant of type m of framed knots then $V \circ \psi^q$ is also a Vassiliev invariant of type m, and that $W_m(V \circ \psi^q) = W_m(V) \circ \psi^q$.

3.2.1. *Proof of part (1) of theorem* 9.

Exercise 3.15. Prove that $\mathcal{W}$ is a sub-Hopf-algebra of $\mathcal{A}^*$ and that $\mathcal{P}(\mathcal{W}) = \mathcal{P}(\mathcal{A}^*) \cap \mathcal{W}$.

The exercise implies that to prove part (1) of theorem 9 it is enough to show that

$$\mathcal{P}(\mathcal{A}^*) \cap \mathcal{W} = \mathcal{P}'(\mathcal{A}^*).$$

A functional $W \in \mathcal{A}^*$ is primitive iff $\Delta(W) = \epsilon \otimes W + W \otimes \epsilon$ iff $W(D_1 \cdot D_2) = \epsilon(D_1)W(D_2) + W(D_1)\epsilon(D_2)$ iff either $D_1 = \bigcirc$ or $D_2 = \bigcirc$. In other words, W is primitive iff it vanishes on reducible diagrams (those diagrams which are a product in a non-trivial way). Now if $W \in \mathcal{P}'(\mathcal{A}^*)$ then W is primitive and $\deg W > 1$. This implies that W vanishes on diagrams having an isolated chord. On the other hand, if $W \in \mathcal{P}(\mathcal{A}^*) \cap \mathcal{W}$, then it vanishes on $\ominus$ and therefore it is of degree higher than 1. $\square$

3.2.2. *Proof of part (2) of theorem* 4. All that that remains to be shown is that there exists a canonical projection

$$(W \to \hat{W}) : (\mathcal{A}^* = \mathcal{S}(\mathcal{P}(\mathcal{A}^*))) \to (\mathcal{W} = \mathcal{S}(\mathcal{P}'(\mathcal{A}^*)))\,.$$

Such a projection is induced from the natural projection $\mathcal{P}(\mathcal{A}^*) \to \mathcal{P}'(\mathcal{A}^*)$ that maps the primitive of degree 1 to 0 and acts as the identity otherwise. $\square$

Exercise 3.16. Define a map $\phi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$\phi(D_1 \otimes D_2) = \begin{cases} \left(-\ominus\right)^{\deg D_1} \cdot D_2 & \text{if } D_1 \text{ is chord diagram,} \\ 0 & \text{if } D_1 \text{ has internal trivalent vertices,} \end{cases}$$

and show that $\hat{W}(D) = W(\phi(\Delta D))$.[10]

[10] I wish to thank S. Willerton for noticing an inaccuracy in my original formulation of this exercise and for supplying this much cleaner version.

4. Kontsevich's Knizhnik-Zamolodchikov construction

In this section we will mostly be concerned with proving the hard part of theorem 1 — with defining the map V. Recall that the 'mth derivatives' of a Vassiliev invariant of type m is a system of 'constants', and that we called such systems of constants *weight systems*. The aim now is to show that every weight system can be 'integrated' m times to give a Vassiliev invariant. An incomplete proof of that fact first appeared in Bar-Natan [6], where it was shown that the combinatorics underlying perturbative Chern-Simons theory is essentially that of weight systems, and that on a naive level, ignoring questions of convergence, perturbative Chern-Simons theory can be used to integrate a weight system to a Vassiliev invariant. The analytical difficulties in [6] were later resolved by Axelrod and Singer [3] and by Kontsevich [26]. Later on, Kontsevich found a second proof based on the Knizhnik-Zamolodchikov equation [23], in which convergence is much easier to show. This is the proof that we will present here.

4.1. Connections, curvature, and holonomy. Up to some (important) subtlety, a connection is a 1-form whose values are in the algebra of endomorphisms of the fiber. One would like to know how much of the theory of connections can be generalized to the case of 1-forms with values in an arbitrary associative algebra. As was shown by K-T. Chen [12], much of the theory persists in the more general case. Let us briefly review some aspects of Chen's theory.

Let X be a smooth manifold and let $\mathfrak{A}$ be a topological algebra over the real numbers $\mathbf{R}$ (or the complex numbers $\mathbf{C}$), with a unit 1. An $\mathfrak{A}$-*valued connection* Ω on X is an $\mathfrak{A}$-valued 1-form Ω on X. Its curvature F_Ω is the $\mathfrak{A}$-valued 2-form $F_\Omega = d\Omega + \Omega\wedge\Omega$, where the definitions of the exterior differentiation operator d and of the wedge product $\wedge$ are precisely the same as the corresponding definitions in the case of matrix valued forms. The notion of "parallel transport" also has a generalization in the new context: Let $B : I \to X$ be a smooth map from some interval $I = [a, b]$ to X. Define the *holonomy* $h_{B,\Omega}$ of Ω along B to be the function $h_{B,\Omega} : I \to \mathfrak{A}$ which satisfies

$$h_{B,\Omega}(a) = 1; \qquad \frac{\partial}{\partial t} h_{B,\Omega}(t) = \Omega\left(\dot{B}(t)\right) h_{B,\Omega}(t), \quad (t \in I)$$

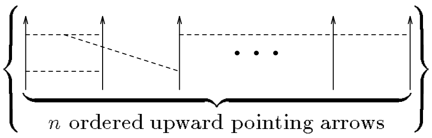
if such a function exists and is unique. In many interesting cases, $h_{B,\Omega}$ exists and is given (see *e.g.* [12]) by the following "iterated integral" formula:

$$h_{B,\Omega}(t) = 1 + \sum_{m=1}^{\infty} \int_{a\leq t_1\leq\ldots\leq t_m\leq t} (B^*\Omega)(t_m)\cdot\ldots\cdot(B^*\Omega)(t_1). \tag{11}$$

(In this formula $B^*\Omega$ denotes the pullback of Ω to I via B). Furthermore, just like in the standard theory of connections, if $F_\Omega \equiv 0$ ('Ω is flat'), then $h_{B,\Omega}$ is invariant under homotopies of B that preserve its endpoints.

In the case of interest for us, $\mathfrak{A}$ will be the completion of a graded algebra of finite type over the complex numbers — the direct product of the finite dimensional (over $\mathbf{C}$) homogeneous components of a graded algebra. The connection Ω will be homogeneous of degree 1. In this case the mth term $h_{m,B,\Omega}$ in (11) is homogeneous of degree m, and there is no problem with the convergence of the sum there. Also, as each term lives in a different degree, Chen's theory implies that each term is invariant under homotopies of B that preserve its endpoints. These assertions are not very hard to verify directly from the definition of $h_{m,B,\Omega}$ as a multiple integral.

4.2. The formal Knizhnik-Zamolodchikov connection. Let $\mathcal{D}_n^{KZ}$ be the collection of all diagrams made of n ordered upward pointing arrows, and dashed arcs and oriented vertices as in the definition of $\mathcal{A}^t$, with the standard conventions about higher than trivalent vertices and about the orientation of vertices:

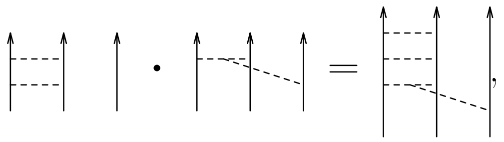
$\mathcal{D}_n^{KZ} =$ 

.

Let the ground field be $\mathbf{C}$ and let $\mathcal{A}_n^{KZ}$ be the quotient

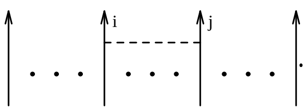
$$\mathcal{A}_n^{KZ} = \operatorname{span}(\mathcal{D}_n^{KZ}) / \{STU \text{ relations}\}\ .$$

$\mathcal{A}_n^{KZ}$ is an algebra with 'composition' as its product:


$(n = 3)$.

$\mathcal{A}_n^{KZ}$ is graded by half the number of vertices in a diagram, excluding the $2n$ endpoints of the n arrows; the degree of the above product is 4.

For $1 \leq i < j \leq n$ define $\Omega_{ij} \in \mathcal{A}_n^{KZ}$ by

$\Omega_{ij} =$ 

.

Let X_n be the configuration space of n distinct points in $\mathbf{C}$; $X_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : z_i = z_j \Rightarrow i = j\}$, and let ω_{ij} be the complex 1-form on X_n defined by

$$\omega_{ij} = d(\log z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}.$$

The formal Knizhnik-Zamolodchikov connection is the $\mathcal{A}_n^{KZ}$-valued connection

$$\Omega_n = \sum_{1 \leq i < j \leq n} \Omega_{ij}\omega_{ij}$$

on X_n.

Proposition 4.1. *The formal Knizhnik-Zamolodchikov connection Ω_n is flat.*

Proof. Clearly $d\Omega_n = 0$. Let us check that

$$\Omega_n \wedge \Omega_n = \sum_{i<j;i'<j'} \Omega_{ij}\Omega_{i'j'}\omega_{ij} \wedge \omega_{i'j'} = 0. \tag{12}$$

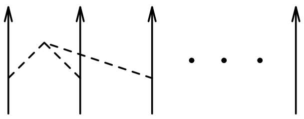
The above sum can be separated into three parts, according to the cardinality of the set $\{i,j,i',j'\}$. If this cardinality is 2 or 4 then Ω_{ij} and $\Omega_{i'j'}$ commute, while ω_{ij} and $\omega_{i'j'}$ anti-commute. It is easy to check that this implies that the corresponding parts of the sum (12) vanish. The only interesting case is when $|\{i,j,i',j'\}| = 3$, say $\{i,j,i',j'\} = \{1,2,3\}$. In this case,

$$\sum_{\{i,j,i',j'\}=\{1,2,3\}} \Omega_{ij}\Omega_{i'j'}\omega_{ij} \wedge \omega_{i'j'} = (\Omega_{12}\Omega_{23} - \Omega_{23}\Omega_{12})\omega_{12} \wedge \omega_{23} + (\text{cyclic permutations}).$$

By the STU relation this is

$$= \Omega_{123}(\omega_{12} \wedge \omega_{23} + (\text{cyclic permutations})) = 0, \tag{13}$$

where Ω_{123} is given by

$\Omega_{123} =$  $\in \mathcal{A}_n^{KZ}$.

The vanishing of $\omega_{12} \wedge \omega_{23} + (\text{cyclic permutations})$ is called 'Arnold's identity' [1] and can be easily verified by a direct computation. □

Remark 4.2. The connection Ω_n has a simple generalization to the case when the underlying algebra is $\mathcal{A}_{n,n}^{KZ}$, the algebra generated by diagrams having $2n$ arrows, whose first n arrows point upward and whose next n arrows point downward. The only difference is a sign difference in the application of the STU relation in (13). Therefore if one defines

$$\Omega_{n,n} = \sum_{1 \le i \le j \le 2n} s_i s_j \Omega_{ij}\omega_{ij},$$

where $s_i = \begin{cases} +1 & i \le n \\ -1 & i > n \end{cases}$, then the connection $\Omega_{n,n}$ is flat.

4.3. Kontsevich's integral invariants. Choose a decomposition $\mathbf{R}^3 = \mathbf{C}_z \times \mathbf{R}_t$ of $\mathbf{R}^3$ to a product of a complex plane $\mathbf{C}_z$ parametrized by z and a real line $\mathbf{R}_t$ parametrized by t and let $K : S^1 \to \mathbf{R}^3$ be a parametrized knot on which the function t is a Morse function. Consider the following series, whose precise definition will be discussed below:

(14)

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{\min} < t_1 < \ldots < t_m < t_{\max}} \sum_{\substack{\text{applicable pairings} \\ P = \{(z_i, z_i')\}}} (-1)^{\# P_{\downarrow}} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \in \mathcal{A}^r_{\mathbf{C}}.$$

In the above equation,

- $t_{\min}$ ($t_{\max}$) is the minimal (maximal) value of t on K.
- an 'applicable pairing' is a choice of an unordered pair (z_i, z_i') for every $1 \leq i \leq m$, for which (z_i, t_i) and (z_i', t_i) are *distinct* points on K.
- $\# P_{\downarrow}$ is the number of points of the form (z_i, t_i) or (z_i', t_i) at which K is decreasing. Remember that in this article we are only considering *oriented* knots.
- $\mathcal{A}^r_{\mathbf{C}}$ is the quotient of $\mathcal{A}_{\mathbf{C}}$ by the ideal generated by the diagram ⊖. In other words, diagrams having an isolated chord (in the sense of definition 1.6) are set equal to 0 in $\mathcal{A}^r_{\mathbf{C}}$. The subscripts $\mathbf{C}$ are intended to remind us that the construction is done over the ground field $\mathbf{C}$. There is a similar definition for $\mathcal{A}^r_{\mathbf{R}}$.
- D_P is the chord diagram naturally associated with K and P as in figure 13. It is to be regarded as an element of the quotient $\mathcal{A}^r_{\mathbf{C}}$.
- Every pairing defines a map $\{t_i\} \mapsto \{(z_i, z_i')\}$ locally around the current values of the t_i's. Use this map to pull the dz_i's and dz_i''s to the m-simplex $t_{\min} < t_1 < \ldots < t_m < t_{\max}$ and then integrate the indicated wedge product over that simplex.

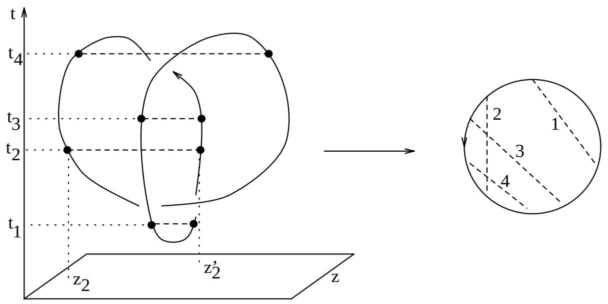
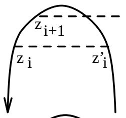
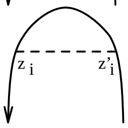


FIGURE 13. $m = 4$: a knot K with a pairing P and the corresponding chord diagram D_P. Notice that $D_P = 0$ in $\mathcal{A}^r_{\mathbf{C}}$ due to the isolated chord marked by 1.

4.3.1. *Finiteness.* Properly interpreted, the integrals in (14) are finite. There appears to be a problem in the denominator when $z_i - z_i'$ is small for some i. This can happen in either of two ways:

(1) 

in this case the integration domain for z_{i+1} is as small as $z_i - z_i'$, and its 'smallness' cancels the singularity coming from the denominator.

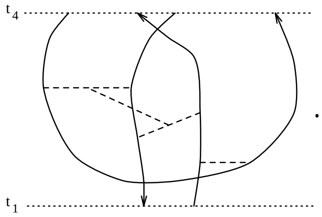
(2) 

in this case the corresponding diagram D_P has an isolated chord, and so it is 0 in $\mathcal{A}^r_{\mathbf{C}}$.

4.3.2. *Invariance under horizontal deformations.* For times $t_{\min} \leq a < b \leq t_{\max}$ define $Z(K, [a, b])$ in exactly the same way as (14), only restricting the domain of integration to be $a < t_1 < \ldots < t_m < b$. Of course, $Z(K, [a, b])$ will not be in $\mathcal{A}^r_{\mathbf{C}}$, but rather in the vector space

$$\mathcal{A}^{K,[a,b]}_{\mathbf{C}} = \operatorname{span}\left\{\begin{matrix}\text{diagrams whose solid lines}\\ \text{are as in the part of } K \text{ on}\\ \text{which } a \leq t \leq b\end{matrix}\right\} \Big/ \operatorname{span}\left\{\begin{matrix}STU \text{ relations and di-}\\ \text{agrams with subdia-}\\ \text{grams like}\end{matrix}\right\}.$$

For example, if t_1, t_4, and K are as in figure 13, then the following is a diagram in $\mathcal{A}^{K,[t_1,t_4]}_{\mathbf{C}}$:



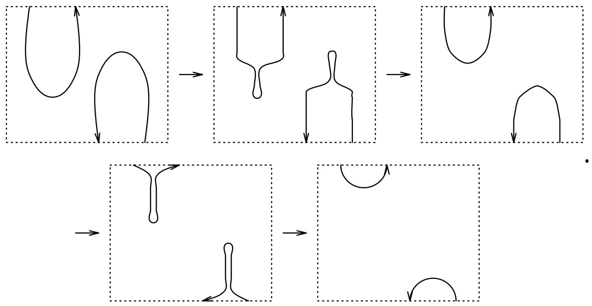
The same reasoning as in section 4.3.1 shows that $Z(K, [a, b])$ is finite. For $t_{\min} \leq a < b < c \leq t_{\max}$, there is an obvious product $\mathcal{A}^{K,[a,b]}_{\mathbf{C}} \otimes \mathcal{A}^{K,[b,c]}_{\mathbf{C}} \to \mathcal{A}^{K,[a,c]}_{\mathbf{C}}$, and it is easy to show that with this product $Z(K, [a, b])Z(K, [b, c]) = Z(K, [a, c])$.

Let $t_{\min} < a < b < t_{\max}$ be times for which K has no critical points in the time slice $a \leq t \leq b$, and let n be the number of upward (or downward) pointing strands of K in that slice. Then $\mathcal{A}^{K,[a,b]}_{\mathbf{C}} \equiv \mathcal{A}^{KZ}_{n,n}$, and comparing with (11) and the definition of $\Omega_{n,n}$ we see that $Z(K, [a, b])$ is the holonomy of $\Omega_{n,n}$ along the braid defined by the intersection of K with the slice $a \leq t \leq b$. The flatness of $\Omega_{n,n}$ implies that this holonomy is invariant under horizontal deformations of that piece of K, and together with

$$Z(K) = Z(K, [t_{\min}, t_{\max}]) = Z(K, [t_{\min}, a])Z(K, [a, b])Z(K, [b, t_{\max}])$$

we see that $Z(K)$ is invariant under horizontal deformations of K which 'freeze' the time slices in which K has a critical point.

4.3.3. *Moving critical points.* In this section we will show that (subject to some restrictions) $Z(K)$ is also invariant under deformations of K that do move critical points. The idea is to narrow critical points to sharp needles using horizontal deformations, and then show that very sharp needles contribute almost nothing to $Z(K)$ and therefore can be moved around (almost) freely:

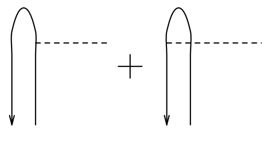
(15) 

Lemma 4.3. *If the two knots $K_{1,2}$ both contain a sharp needle of width ϵ, and are the identical except possibly for the length and the directions of their respective needles, then*

$$||Z_m(K_1) - Z_m(K_2)|| \sim \epsilon$$

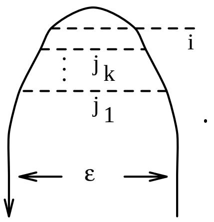
where Z_m is the degree m piece of Z and $||\cdot||$ is some fixed norm on $\mathcal{G}_m\mathcal{A}^r_{\mathbf{C}}$.

Proof. Clearly, the difference between $Z_m(K_1)$ and $Z_m(K_2)$ will come only from terms in (14) in which one of the z_i's (or z_i''s) is on the needle. So let us show that if a knot K contains a needle N of width ϵ, then such terms in $Z_m(K)$ are at most proportional to ϵ. Without loss of generality we can assume that the needle N points upward. If the highest pair (z_i, z_i') that touches N connects the two sides of N, the corresponding diagram is 0 in $\mathcal{A}^r_{\mathbf{C}}$ and there is nothing to worry about. If there is no pair (z_j, z_j') that connects the two sides of N then again life is simple: in that case there are no singularities in (14) so nothing big prevents



from being small. (Notice that these two terms appear in $Z(K)$ with opposite signs due to the factor $(-1)^{\#P_\downarrow}$ but otherwise they differ only by something proportional to ϵ). If (z_j, z_j') is a pair that does connect the two sides of N, it has to do so in the top (round) part of N — otherwise $dz_j - dz_j' = 0$.

So the only terms that cause some worry are those that have some $k > 1$ pairs $(z_{j_1}, z'_{j_1}), \ldots, (z_{j_k}, z'_{j_k})$ on the top part of N, with (z_{j_k}, z'_{j_k}) being the highest of these pairs and (z_{j_1}, z'_{j_1}) the lowest. We might as well assume that there are no pairs other than (z_i, z_i') that touch N only once — such pairs just shorten the domain of integration in (14) without adding any singularity in the denominator. So what we have looks like:

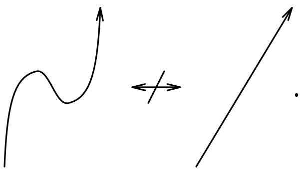
(16) 

.

Writing $\delta_\alpha = |z_{j_\alpha} - z'_{j_\alpha}|$, we see that the integral corresponding to (16) is bounded by a constant times

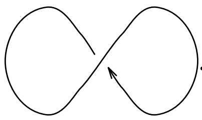
$$\int_0^\epsilon \frac{d\delta_1}{\delta_1} \int_0^{\delta_1} \frac{d\delta_2}{\delta_2} \cdots \int_0^{\delta_{k-1}} \frac{d\delta_k}{\delta_k} \int_{z_{j_k}}^{z'_{j_k}} \frac{dz_i - dz'_i}{z_i - z'_i} \sim \epsilon. \quad \square$$

Unfortunately, there is one type of deformation that (15) and lemma 4.3 cannot handle — the total number of critical points in K cannot be changed:

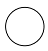
(17) 
.

Even if the hump on the left figure is deformed into a needle and then this needle is removed, a (smaller) hump still remains.

4.3.4. *The correction.* Let the symbol ∞ stand for the embedding


.

Notice that

(18) $$Z(\infty) = \text{}$$  $+$ (higher order terms)

and so using power series $Z(\infty)$ can be inverted and the following definition makes sense:

Definition 4.4. Let K be a knot embedded in $\mathbf{C} \times \mathbf{R}$ with c critical points. Notice that c is always even and set[11]

$$\tilde{Z}(K) = \frac{Z(K)}{(Z(\infty))^{\frac{c}{2}-1}}.$$

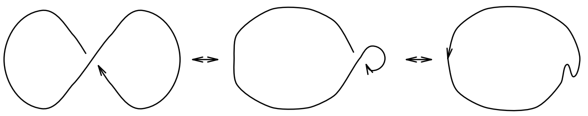
Theorem 13. *$\tilde{Z}(K)$ is invariant under arbitrary deformations of the knot K.*

[11]The non-invariance of $Z(K)$ under the move (17) was first noticed by R. Bott and the author. The correction $\tilde{Z}(K)$ is due to Kontsevich [25].

Proof. Clearly, $\tilde{Z}(K)$ is invariant under deformations that do not change the number of critical points of K, and the only thing that remains to be checked is its invariance under the move (17). So let K_c and K_s be two knots that are identical other then that in some place K_c has the figure in the left side of (17) while in the same place K_s has the figure on the right side of (17). We need to show that in $\mathcal{A}^r_{\mathbf{C}}$,

$$Z(K_c) = Z(\infty)Z(K_s).$$

Using deformations as in section 4.3.3 we can move the 'humps' of K_c to be very far from the rest of the knot, and shrink them to be very small. This done, we can ignore contributions to $Z(K_c)$ coming from pairings in which any of the pairs connect the humps to the rest of the knot. Hence $Z(K_c)$ factors to a part which is the same as in $Z(K_s)$ times contributions that come from pairings that pair the 'humpy' part of K_c to itself. But as the following figure shows, for the same reasons as in section 4.3.3, these contributions are precisely $Z(\infty)$:

 $\square$

Exercise 4.5. Show that $\tilde{Z}(K)$ is in fact real: $\tilde{Z}(K) \in \mathcal{A}^r_{\mathbf{R}}$.

Hint 4.6. Use the fact that the transformation $t \to -t$, $z \to \bar{z}$ maps a knot to an equivalent knot, while mapping $\Omega_{n,n}$ to minus its conjugate.

Remark 4.7. Kontsevich [25], building on some work of Drinfel'd ([16] and [17]), has proven that $\tilde{Z}(K)$ has rational coefficients.

4.4. Proof of theorem 1.

4.4.1. *Proof of part (2) of theorem* 1. A weight system W of degree m is just a linear functional on $\mathcal{G}_m\mathcal{A}^r_{\mathbf{R}}$. Extend it by zero to all of $\mathcal{A}^r_{\mathbf{R}}$, and define

$$V(W)(K) = W(\tilde{Z}(K)). \quad \square$$

4.4.2. *Proof of part (3) of theorem* 1. Let W be a degree m weight system. To show that $W = W_m(V(W))$ it is enough to show that if $D \in \mathcal{G}_m\mathcal{D}^c$ is a chord diagram of degree m and if K_D is an embedding of D in the sense of section 2.2, then (for the natural extension of $\tilde{Z}$ to knots with double points, definition 1.2):

$$\tilde{Z}(K_D) = \bar{D} + (\text{terms of degree} > m),$$

where $\bar{D}$ is the class of D in $\mathcal{A}^r_{\mathbf{R}}$. In view of (18), it is enough to prove the same for Z rather than for $\tilde{Z}$. If two knots K^o and K^u are identical except that two of their strands form an overcrossing in K^o and an undercrossing in K^u, it is clear that the only contributions to $Z(K^o) - Z(K^u)$ come from pairings in which these two strands are paired. $Z(K_D)$ is a signed sum of Z evaluated on 2^m knots, and this sum can be partitioned in pairs like the above $K^{o,u}$ around m different crossings —

and thus contributions to $Z(K_D)$ come only from pairings that pair the strands near any of the m double points of K_D. This implies that the lowest degree contribution to $Z(K_D)$ is at least of degree m. In degree m the pairing P is determined by the above restriction. It is easy to see that in that case $D_P = D$, and therefore the piece of degree precisely m in $Z(K_D)$ is proportional to $\bar{D}$. It remains to determine the constant of proportionality. This is a simple computation — in degree 1, the difference between $Z(K^o)$ and $Z(K^u)$ comes from the difference between integrating

$$\frac{dz - dz'}{z - z'}$$

along a contour in which z passes near but above z' and along a contour in which z passes near but under z'. By Cauchy's theorem this is $2\pi i$. Repeating this m times for each of the m double points of K_D, we get $(2\pi i)^m$ and this exactly cancels the $(2\pi i)^m$ in the denominator of (14). This proves that $W = W_m(V(W))$.

Next, let V be a Vassiliev invariant of type m. By the above discussion,

$$W_m(V - V(W_m(V))) = W_m(V) - W_m(V(W_m(V))) = W_m(V) - W_m(V) = 0.$$

In other words, V and $V(W_m(V))$ differ by an invariant whose mth derivative vanishes. Namely, by an invariant of type $m - 1$. $\square$

Remark 4.8. The argument in section 4.4.2 together with Kohno's theorem [24] immediately imply theorem 5, as was observed by Piunikhin [33].

Problem 4.9. Is Kontsevich's construction the same as the Reshetikhin-Turaev construction? In other words, consider the diagram:

$$\begin{tikzcd} & \{\mathfrak{g}, R, m\} \arrow[dl, "\text{theorem 4}"'] \arrow[dr, "\text{theorem 5}"] & \\ \mathcal{W} \arrow[rr, shift left, "\text{theorem 1}"'] & & \mathcal{V} \arrow[ll, shift left] \end{tikzcd} \tag{19}$$

Theorem 5 says that the two ways of getting from a triple $\{\mathfrak{g}, R, m\}$ to $\mathcal{W}$ are the same. Is it also true that the two ways of getting from such a triple to $\mathcal{V}$ are the same? Notice that in the Reshetikhin-Turaev construction $\mathfrak{g}$-invariance is broken by a choice of a Cartan sub-algebra (and then the result is shown not to depend on that choice), whereas in Kontsevich's construction (definition 4.4) $\mathfrak{g}$-invariance is never broken. This means that if the two constructions do agree, then Kontsevich's is perhaps somewhat "better". *Warning:* Low order computations show that for diagram (19) to commute, $\tilde{Z}$ should be redefined to be $Z(K)/Z(\infty)^{c/2}$. This does not change the conclusions of theorems 13 and 1, but does invalidate theorem 9. It appears that [16, 17] imply a positive answer to this problem, but the details of such an argument are yet to be carefully checked.

Problem 4.10. Is Kontsevich's construction the same as the Chern-Simons construction alluded to in the beginning of section 4?

Problem 4.11. The statement of theorem 1 is purely combinatorial, and one would expect that the proof would also be combinatorial. I see the analytical proof given here as a temporary argument until a combinatorial proof will be found. See also problem 7.3.

5. The primitive elements of $\mathcal{A}$

5.1. Theorem 8 from the perspective of Lie algebras. One of the versions of the Poincare-Birkhoff-Witt (PBW) theorem (see *e.g.* [15, sec. 2.4.10]) says that the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is canonically isomorphic to the symmetric algebra $\mathfrak{S}(\mathfrak{g})$ on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on the two spaces in a compatible way, and therefore the $\mathfrak{g}$-invariant pieces of these two spaces are isomorphic:

$$\mathfrak{U}(\mathfrak{g})^{\mathfrak{g}} \simeq \mathfrak{S}(\mathfrak{g})^{\mathfrak{g}}. \tag{20}$$

The next exercise shows that in some sense the space $\mathcal{A}$ corresponds to $\mathfrak{U}(\mathfrak{g})^{\mathfrak{g}}$ and the space $\mathcal{B}$ corresponds to $\mathfrak{S}(\mathfrak{g})^{\mathfrak{g}}$. In the light of (20) it is therefore not too surprising to find that $\mathcal{A} \simeq \mathcal{B}$.

Exercise 5.1. Given a Lie algebra $\mathfrak{g}$ and a metric t construct natural maps $\mathcal{T}_{\mathcal{A}} : \mathcal{A}^l \to \mathfrak{U}(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{T}_{\mathcal{B}} : \mathcal{B} \to \mathfrak{S}(\mathfrak{g})^{\mathfrak{g}}$. Notice that these maps do not preserve the natural gradings of the spaces involved.

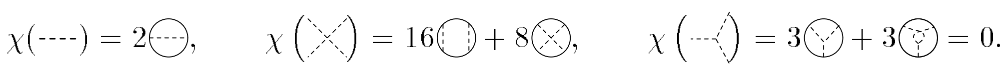
Hint 5.2. For $\mathcal{T}_{\mathcal{A}}$ carry out the same procedure as in section 2.2, ignoring those parts of the procedure that mention the representation R. Given $D \in \mathcal{D}^l$ get a tensor in some tensor power of $\mathfrak{g}$, and use the product of $\mathfrak{U}(\mathfrak{g})$ to land in $\mathfrak{U}(\mathfrak{g})^{\mathfrak{g}}$. The map $\mathcal{T}_{\mathcal{B}}$ is even easier, only that there is no natural ordering for the external vertices of a diagram $C \in \mathcal{C}$.

Problem 5.3. The maps $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{B}}$ are not isomorphisms (see *e.g.* (30)), but I do not know how far they are from being isomorphisms.

Problem 5.4. Exercise 5.1 shows that to get a linear functional on $\mathcal{A}^l$ it is sufficient to choose a linear functional on the center of $\mathfrak{U}(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$ that has an invariant metric. If $\mathfrak{g}$ is semi-simple, it is well known that the space of such linear functionals is spanned by traces in finite dimensional representations and thus linear functionals on $\mathfrak{U}(\mathfrak{g})^{\mathfrak{g}}$ give the same $\mathcal{T}$'s as in section 2.4. I do not know if this is still true if $\mathfrak{g}$ is not semi-simple.

The proof of theorem 8 is more or less a direct translation of the proof of the above version of the PBW theorem to our language. It is instructive to read the following proof keeping Lie algebras in mind and noticing that $\mathcal{T}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{B}}$ carry the isomorphism that we construct to the standard isomorphism (20).

5.2. Proof of theorem 8. We will construct a map $\bar{\chi} : \mathcal{B} \to \mathcal{A}$ and a map $\bar{\sigma} : \mathcal{A} \to \mathcal{B}$, and then show that they are each other's inverse. For a Chinese character $C \in \mathcal{C}$ that has k external vertices, define $\chi(C)$ to be the sum of the $k!$ ways of enclosing C in a circle. Each of these $k!$ ways is obtained by choosing a bijection between the external vertices of C and the collection of kth roots of unity $\{z : z^k = 1\}$, and then gluing each external vertex of C to the corresponding point on the unit circle. For example,



Theorem 6 implies that χ descends to a map $\bar{\chi} : \mathcal{B} \to \mathcal{A}$.

The idea of the proof is to try to invert $\bar{\chi}$. The image of $\bar{\chi}$ are the *symmetric* linear combinations of diagrams — those combinations that are stable under permuting external vertices. So given any diagram we will try to show that it is equivalent (modulo STU relations) to a symmetric combination. The obvious way to do that is to *symmetrize* — to compare each diagram with the sum of all of its 'permuted versions'. We will show that the difference between a diagram and a permuted version of it is equivalent (mod STU) to a sum of diagrams with a lower number of external vertices (but this sum is *not* uniquely determined), and so we can 'push the problems down' and prove by induction. The most central of the technical details that have to be checked is the uniqueness of the procedure outlined above. This is where the properties defining $\mathcal{B}$, the IHX relation and anti-symmetry of internal vertices, are used.

To define the map $\bar{\sigma}$ we first need to filter $\mathcal{D}$ and $\mathcal{C}$. Let

$$\mathcal{D}_k = \operatorname{span}\{D \in \mathcal{D}^l : D \text{ has at most } k \text{ external vertices}\},$$

$$\mathcal{C}_k = \operatorname{span}\{C \in \mathcal{C} : C \text{ has at most } k \text{ external vertices}\}.$$

Let $\mathcal{I}_k$ be the ideal in $\mathcal{D}$ generated by all STU relations that do not involve diagrams with more than k external vertices (so that $\mathcal{A} = \mathcal{D}/\mathcal{I}_\infty$).

We will inductively construct a sequence of compatible maps $\sigma_k : \mathcal{D}_k \to \mathcal{B}$ satisfying:

($\Sigma 1$) $\sigma_k \circ \chi_k = I_{\mathcal{C}_k}$ where χ_k is the restriction of χ to $\mathcal{C}_k$ and $I_{\mathcal{C}_k}$ is the identity map of $\mathcal{C}_k$.

($\Sigma 2$) σ_k(the anti-symmetry relation) $= 0$. In other words, if the two diagrams $D_\pm \in \mathcal{D}_k$ differ only by the orientation of one of their vertices, then $\sigma_k(D_+ + D_-) = 0$.

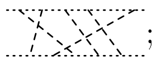
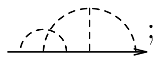
($\Sigma 3$) $\sigma_k(IHX) = 0$. In other words, if the diagrams $I, H, X \in \mathcal{D}_k$ are related as in theorem 6 then $\sigma_k(I - H + X) = 0$.

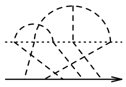
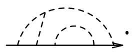
($\Sigma 4$) $\sigma_k(\mathcal{I}_k) = 0$. (And thus σ_k descends to a map $\bar{\sigma}_k : \mathcal{D}_k/\mathcal{I}_k \to \mathcal{B}$).

($\Sigma 5$) $\bar{\chi} \circ \bar{\sigma}_k = I_{\mathcal{D}_k/\mathcal{I}_k}$.

For $k = 0$ there is a single diagram in $\mathcal{D}_0$ (that is, $\bigcirc$) and single diagram in $\mathcal{C}_0$ — the empty diagram E. So define $\sigma_0\left(\bigcirc\right) = E$. A diagram $D \in \mathcal{D}_1$ has exactly one vertex on its circle. Define $\sigma(D)$ to be D with its circle removed and then $\Sigma 1$-$\Sigma 3$ is trivial. $\mathcal{I}_1$ is empty and so for $\Sigma 4$ there is nothing to prove and $\Sigma 5$ is trivial. Let us assume for some $k > 1$ a map σ_{k-1} satisfying $\Sigma 1$-$\Sigma 5$ was constructed.

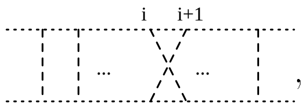
The permutation group $\mathcal{S}_k$ acts on diagrams in $\mathcal{D}^l$ having precisely k external vertices by permuting those external vertices. This operations has a nice graphic representation as 'composition':

$k = 5$; $\pi =$  ; $D =$  ;

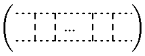
(21) $\pi D =$  $= -$  .

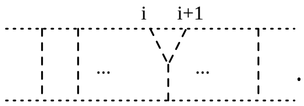
Lemma 5.5. *Regarded as an element of $\mathcal{D}_k/\mathcal{I}_k$, $D - \pi D$ is an the image of $\beta : \mathcal{D}_{k-1} \to \mathcal{D}_k \to \mathcal{D}_k/\mathcal{I}_k$. Any choice U of a presentation of π as a product of transpositions determines in a natural way an element $\Gamma_D(U) \in \mathcal{D}_{k-1}$ for which $\beta(\Gamma_D(U)) = D - \pi D$. Furthermore, if U and $\bar{U}$ are two such presentations, then $\Lambda_D(U) = \Lambda_D(\bar{U})$, where $\Lambda_D \overset{def}{=} \sigma_{k-1} \circ \Gamma_D$.*

Proof. For $1 \leq i < k$ let $U_i = (i, i+1)$ be the transposition that interchanges i and $i+1$,

$U_i =$ 

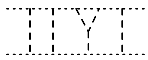
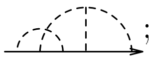
,

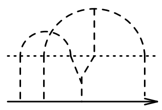
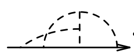
let T denote the identity () in $\mathcal{S}_k$, and let S_i be the graph

$S_i =$ 

.

Regard S_i as an operator $S_i : \mathcal{D}_k \to \mathcal{D}_{k-1}$ by composition (as in (21)),

$k = 5$; $S_3 =$  ; $D =$  ;

$S_3 D =$  $=$  .

The STU relation is just the fact that in $\mathcal{D}_k/\mathcal{I}_k$,

(22) $$S_i D = TD - U_i D, \qquad 1 \leq i < k$$

Let U denote a presentation $\pi = U_{i_1} \cdot \ldots \cdot U_{i_\mu}$ of π as a product of transpositions. Set

(23) $$\Gamma_D(U) = \sum_{\nu=1}^{\mu} S_{i_\nu} U_{i_{\nu+1}} \cdot \ldots \cdot U_{i_\mu} D \in \mathcal{D}_{k-1}.$$

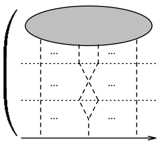
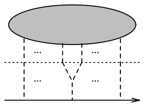
Using (22) on $\beta(\Gamma_D(U))$ we get a telescopic series whose sum is $D - \pi D$.

What we have constructed above is, in fact, a map $\Gamma_D : U \mapsto \Gamma_D(U)$ from the free monoid $\mathcal{M}$ on $k-1$ letters $\{U_1, \ldots, U_{k-1}\}$ (with T denoting its identity) to $\mathcal{D}_{k-1}$. $\mathcal{D}_{k-1}$ is an Abelian group and so this map has an extension to a linear map $\Gamma_D : \mathcal{R}(\mathcal{M}) \to \mathcal{D}_{k-1}$ defined on the monoid-ring $\mathcal{R}(\mathcal{M})$ of $\mathcal{M}$. To conclude the proof of the lemma we need to show that Λ_D vanishes on the kernel $\mathcal{K}$ of the natural map $\mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{S}_k)$. The kernel $\mathcal{K}$ is the double-sided ideal of $\mathcal{R}(\mathcal{M})$ generated by

$$\left\{U_i^2 - T\right\} \cup \{U_iU_j - U_jU_i : |i-j| > 1\} \cup \{U_iU_{i-1}U_i - U_{i-1}U_iU_{i-1} : 1 < i < k\}.$$

Let us first show that Λ_D vanishes on these generators.

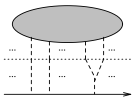
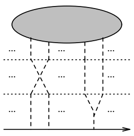
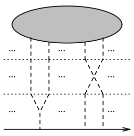
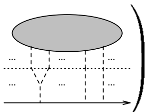
Generators of the form $U_i^2 - T$:

$$\Lambda_D(U_i^2 - T) = \sigma_{k-1}(S_iU_iD + S_iD) = \sigma_{k-1}\Bigg($$  $+$  $$\Bigg) = 0$$

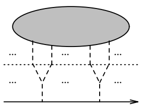
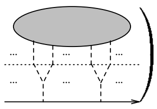
by the anti-symmetry of the vertex ($\Sigma2$).

Generators of the form $U_iU_j - U_jU_i$, $|i-j| > 1$:

$$\Lambda_D(U_iU_j - U_jU_i) = \sigma_{k-1}(S_jD - S_jU_iD + S_iU_jD - S_iD)$$

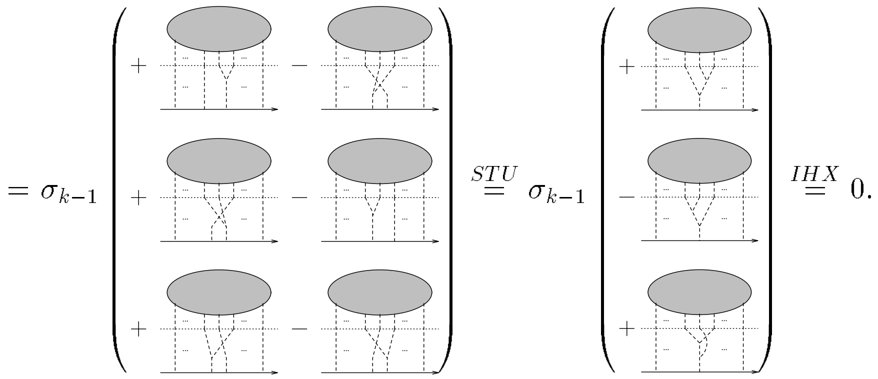
$$= \sigma_{k-1}\Bigg($$  $-$  $+$  $-$  $$\Bigg)$$

by the STU relation ($\Sigma4$) this is

$$= \sigma_{k-1}\Bigg($$  $-$  $$\Bigg) = 0.$$

Generators of the form $U_iU_{i-1}U_i - U_{i-1}U_iU_{i-1}$:

$$\Lambda_D(U_iU_{i-1}U_i - U_{i-1}U_iU_{i-1}) = \sigma_{k-1}\begin{pmatrix} +S_iD & -S_{i-1}U_iU_{i-1}D \\ +S_iU_{i-1}U_iD & -S_{i-1}D \\ +S_{i-1}U_iD & -S_iU_{i-1}D \end{pmatrix}$$



Here the IHX identity ($\Sigma 3$) was used to deduce the last equality, and the STU identity ($\Sigma 4$) was used separately in each row to deduce the preceding equality.

Having shown that Λ_D vanishes on the generators of $\mathcal{K}$, it is rather easy to conclude that Λ_D vanishes identically on $\mathcal{K}$. Notice that all the generators of $\mathcal{K}$ are of the form $W_1 - W_2$ for some $W_{1,2} \in \mathcal{M}$. Notice also that (23) implies that Λ has the following 'cocycle property':

$$\Lambda_D(VW) = \Lambda_{WD}(V) + \Lambda_D(W), \qquad (V, W \in \mathcal{M};\ D \in \mathcal{D}_k). \tag{24}$$

Therefore, if VWZ is a general element of $\mathcal{K}$ where $W = W_1 - W_2$ is one of the generators considered above, then

$$\begin{aligned}\Lambda_D(VW_1Z - VW_2Z) =& (\Lambda_{W_1ZD}(V) - \Lambda_{W_2ZD}(V)) + (\Lambda_{ZD}(W_1) - \Lambda_{ZD}(W_2)) \\ &+ (\Lambda_D(Z) - \Lambda_D(Z)).\end{aligned} \tag{25}$$

In (25) the first term vanishes because $W_1 - W_2 \in \mathcal{K}$ implies that $W_1 = W_2$ as permutation in $\mathcal{S}_k$ and thus $W_1ZD = W_2ZD$, the middle term vanishes as was shown above and the last term vanishes trivially. $\square$

Let us return to the construction of σ_k and the proof of theorem 8. Lemma 5.5 means that given any $D \in \mathcal{D}_k$ and $\pi \in \mathcal{S}_k$ we can set $\Lambda_D(\pi) = \sigma_{k-1}(\Gamma_D(U_\pi))$ where U_π is *any* presentation of π. Given any diagram $D \in \mathcal{D}_k$ let D^{CC} be the Chinese character obtained from D by removing its directed line, and set

$$\sigma_k(D) = \frac{1}{k!}\left(D^{CC} + \sum_{\pi \in \mathcal{S}_k} \Lambda_D(\pi)\right).$$

For a diagram D having less than k external vertices, just set $\sigma_k(D) = \sigma_{k-1}(D)$.
Proof of $\Sigma 1$: Any Chinese character $C \in \mathcal{C}_k$ is of the form $C = D^{CC}$ for some $D \in \mathcal{D}_k$, and clearly $\chi(C) = \sum_\rho \rho D$. Therefore

$$\sigma_k(\chi_k(C)) = \sigma_k\left(\sum_{\rho \in \mathcal{S}_k} \rho D\right) = \frac{1}{k!}\left(\sum_{\rho \in \mathcal{S}_k} (\rho D)^{CC} + \sum_{\rho, \pi \in \mathcal{S}_k} \Lambda_{\rho D}(\pi)\right)$$

by the cocycle property (24) this is

$$= D^{CC} + \frac{1}{k!}\sum_{\rho,\pi\in\mathcal{S}_k}(\Lambda_D(\pi\rho)-\Lambda_D(\rho)) = C + \sum_{\lambda\in\mathcal{S}_k}\Lambda_D(\lambda) - \sum_{\rho\in\mathcal{S}_k}\Lambda_D(\rho) = C.$$

Proof of $\Sigma 2$ and $\Sigma 3$: If the two diagrams $D_\pm\in\mathcal{D}_k$ differ only by the orientation of one of their internal vertices, then clearly D_+^{CC} and D_-^{CC} also differ only by such an orientation. Hence $\Sigma 2$ follows from the anti-symmetry of internal vertices in $\mathcal{B}$, from the fact (23) that $\Gamma_{D_\pm}$ does not touch the internal vertices of $D_\pm$, and from the induction hypothesis. The proof of $\Sigma 3$ is similar.

Proof of $\Sigma 4$: Using the notation of (22), we need to show that $\sigma_k(S_iD)=\sigma_k(TD-U_iD)$ for a diagram $D\in\mathcal{D}_k$:

$$\sigma_k(TD)-\sigma_k(U_iD) = \frac{1}{k!}\left(D^{CC}-(U_iD)^{CC}+\sum_{\pi\in\mathcal{S}_k}(\Lambda_D(\pi)-\Lambda_{U_iD}(\pi))\right)$$

using the cocycle property (24) this is

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}(\Lambda_D(\pi)-\Lambda_D(U_i\pi)+\Lambda_D(U_i)) = \Lambda_D(U_i) = \sigma_{k-1}(\Gamma_D(U_i)) = \sigma_k(S_iD).$$

Proof of $\Sigma 5$: Let $D\in\mathcal{D}_k$ be regarded as an element of $\mathcal{D}_k/\mathcal{I}_k$. Then

$$\bar\chi(\bar\sigma_k(D)) = \frac{1}{k!}\bar\chi\left(D^{CC}+\sum_{\pi\in\mathcal{S}_k}\Lambda_D(\pi)\right) = \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}(\pi D + (\bar\chi\circ\bar\sigma_{k-1})(\Gamma_D(U_\pi)))$$

using the induction hypothesis ($\Sigma 5$) and then lemma 5.5 this is

$$=\frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}(\pi D+\Gamma_D(U_\pi)) = \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k}(\pi D + D - \pi D) = D.$$

This concludes the proof of the equivalence of $\mathcal{A}$ and $\mathcal{B}$.

Let $C=\bigcup_{i\in I}C_i$ be a presentation of a diagram $C\in\mathcal{C}$ as the union of its connected components. Define a co-product $\Delta_\mathcal{C}:\mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ by

$$\Delta_\mathcal{C}(C) = \sum_{J\subseteq I}\left(\bigcup_{i\in J}C_i\right)\otimes\left(\bigcup_{i\in I\setminus J}C_i\right).$$

It is easy to verify that

$$(\chi\otimes\chi)\circ\Delta_\mathcal{C} = \Delta\circ\chi. \tag{26}$$

This implies that under the equivalence of $\mathcal{A}$ and $\mathcal{B}$, $\Delta_\mathcal{C}$ corresponds to the co-product of $\mathcal{A}$. From the definition of $\Delta_\mathcal{C}$ it is easy to check that the primitive elements of the co-algebra $\mathcal{B}$ are the equivalence classes of connected diagrams in $\mathcal{C}$. This concludes the proof of theorem 8. $\square$

Remark 5.6. The statement of theorem 8 is due to M. Kontsevich [25], as well as the definition of the map χ. The above proof is due to the author.

Remark 5.7. The product $\cdot$ on $\mathcal{A}$ *does not* (!) correspond to taking the union of diagrams in $\mathcal{C}$. Using disjoint union to make $\mathcal{B}$ an algebra, this means that the maps $\mathcal{A} \leftrightarrow \mathcal{B}$ are vector space isomorphisms but not algebra isomorphisms. Pulling back to $\mathcal{A}$ the product of $\mathcal{B}$, we see that $\mathcal{A}$ *is an algebra in two different ways*, both compatible with its co-product.

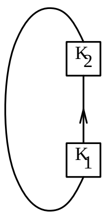
Exercise 5.8. Show that in the case of Lie algebras, the proof of theorem 8 can be changed in a minor way to yield a proof of the full PBW theorem

$$\mathfrak{U}(\mathfrak{g}) \simeq \mathfrak{S}(\mathfrak{g})$$

The idea is to allow an additional type of univalent vertices in $\mathcal{A}^l$ and $\mathcal{B}$, marked by elements of $\mathfrak{g}$. Then mod out by a few more relations, and repeat essentially the same argument as above. Notice that, by a small further modification, this can be done even if the Lie algebra $\mathfrak{g}$ does not carry a metric.

5.3. Proof of part (2) of theorem 9.

Let K_1 and K_2 be knots, and let $K_1\#K_2$ denote their connected sum. Draw $K_1\#K_2$ as on the right, and use the same argument as in the proof of theorem 13 to show that $Z(K_1\#K_2) = Z(K_1)Z(K_2)$, and therefore, after carefully counting critical points in $K_1\#K_2$, $\tilde{Z}(K_1\#K_2) = \tilde{Z}(K_1)\tilde{Z}(K_2)$. If W is a primitive weight system, it satisfies $\Delta(W) = W \otimes \epsilon + \epsilon \otimes W$, and then by the definition of Δ on $\mathcal{A}^*$ we get:



$$\begin{aligned} V(W)(K_1\#K_2) &= W(\tilde{Z}(K_1\#K_2)) = W(\tilde{Z}(K_1)\tilde{Z}(K_2)) = \Delta(W)(\tilde{Z}(K_1) \otimes \tilde{Z}(K_2)) \\ &= W(\tilde{Z}(K_1))\epsilon(\tilde{Z}(K_2)) + \epsilon(\tilde{Z}(K_1))W(\tilde{Z}(K_2)) \\ &= V(W)(K_1) + V(W)(K_2). \quad \square \end{aligned}$$

6. How big are $\mathcal{A}$, $\mathcal{W}$ and $\mathcal{P}$?

6.1. Numerical results. 2,106 lines of C++ code and about 10 days of *CPU* time yield the following results for the dimensions of the various spaces involved [7]:

m	0	1	2	3	4	5	6	7	8	9
$\dim \mathcal{G}_m\mathcal{A}$	1	1	2	3	6	10	19	33	60	104
$\dim \mathcal{G}_m\mathcal{W}$	1	0	1	1	3	4	9	14	27	44
$\dim \mathcal{G}_m\mathcal{P}$	1	1	1	1	2	3	5	8	12	18
$\dim \mathcal{G}_m Lie$	1	1	2	3	6	10	19	33	60	104
diagrams	1	0	1	2	7	36	300	3,218	42,335	644,808
relations	0	0	0	2	15	144	1,645	21,930	334,908	5,056,798

Remark 6.1.
- The space $\mathcal{G}_m Lie$ is the subspace of $(\mathcal{G}_m\mathcal{A})^*$ spanned by the Lie algebraic weight systems of theorem 4. In fact, I have only computed the weight systems corresponding to the classical groups and their representations, and so the above numbers also prove conjecture 2 up to degree 9.
- The numbers in the last two rows indicate the size of the matrices that had to be row reduced in order to compute $\dim \mathcal{G}_m\mathcal{W}$. These numbers have no real significance — they somewhat depend on the details of the algorithm chosen — and are displayed only so as to give an impression of the complexity of the problem.
- The dimensions of $\mathcal{G}_m\mathcal{A}$ and $\mathcal{G}_m\mathcal{P}$ were deduced from $\dim \mathcal{G}_m\mathcal{W}$ using equation (5) and theorem 9.
- I wish to thank V. I. Arnold for correcting an earlier mistake I made in computing $\dim \mathcal{G}_8\mathcal{P}$.
- The problem is highly exponential and it is unlikely that it will be possible to use the same techniques to compute $\dim \mathcal{G}_{10}\mathcal{A}$.

The last point makes it evident that a computer search is not the best way to generate weight systems. In the next section, I will present a very general construction of weight systems, which conceivably generates all of them.

6.2. Marked surfaces. In this section, we will construct the map $\Phi : \mathcal{B} \to \mathcal{M}$, promised in theorem 10. The easiest way to do so, is to factor it through the space $\mathcal{B}^m$ of equivalence classes of *marked diagrams*.

Definition 6.2. A *marked diagram* is a Chinese character (see definition 1.11) whose arcs are marked either by the symbol '=' or by the symbol '×'. Sometimes we will mark an arc with more than one symbol. In this case, it is understood that an even number of ×'s (and an arbitrary number of ='s) on a single arc is equivalent to a single '=' on that arc, while an odd number of ×'s (and an arbitrary number of ='s) is equivalent to a single '×'. (See figure 14). The collection of all marked diagrams is denoted by $\mathcal{C}^m$ and is graded in the same way as $\mathcal{C}$.

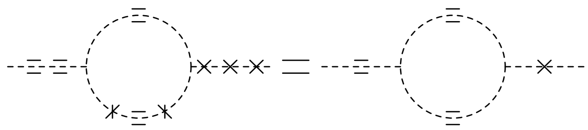
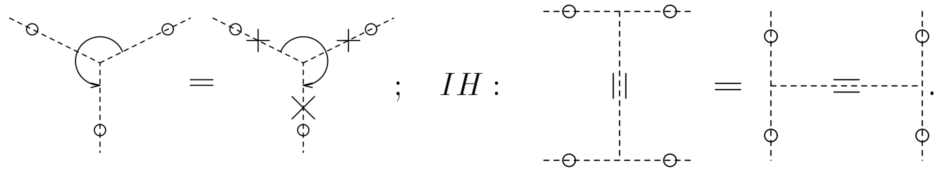


FIGURE 14. Two versions of the same degree 2 marked diagram.

Definition 6.3. $\mathcal{B}^m$ is the quotient of $\operatorname{span}\mathcal{C}^m$ by the subspace spanned by the relations:

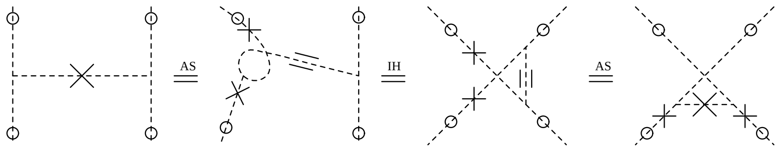
AS : ; IH :



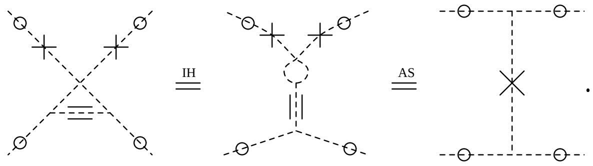
In these marked diagrams the symbol '$\circ$' stands for an arbitrary additional marking, which is the same on corresponding arcs on both sides of the same equation.

Notice that the following two additional relations are satisfied in $\mathcal{B}^m$:

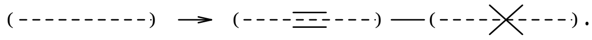
HX :



XI :



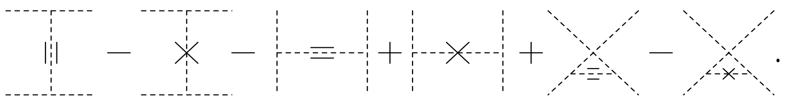
Definition 6.4. Let $\mu : \mathcal{C} \to \mathcal{B}^m$ be the map defined by



In other words, a diagram C in $\mathcal{C}$ having e edges is mapped to the (signed) sum of the 2^e ways of marking C.

Proposition 6.5. *The map μ descends to a map $\mu : \mathcal{B} \to \mathcal{B}^m$.*

Proof. Let us show that $\mu(I - H + X) = 0$. By definition, $\mu(I - H + X)$ is



This sum is equal to 0 in $\mathcal{B}^m$ as can be seen easily from the IH, HX, and XI identities. The marking of the external arcs of I, H, and X is also summed over, and thus the additional $\times$'s on some of the external arcs in the HX and the XI identities do not cause any trouble. Notice that these additional $\times$'s come in pairs and therefore there is no difficulty with signs. A similar check proves that $\mu(C_+ - C_-) = 0$ if C_- differs from C_+ only by the orientation of a single vertex, and together with $I - H + X = 0$ these are precisely the relations that define $\mathcal{B}$. $\square$

Definition 6.6. Define the *thickening* map $\tau : \mathcal{C}^m \to \mathcal{M}$ by simply thickening each of the arcs of a diagram $C \in \mathcal{C}^m$, with or without a twist according to whether or not it is marked by a '$\times$', adding a counterclockwise tangent to the boundary near every external vertex (see figure 15).

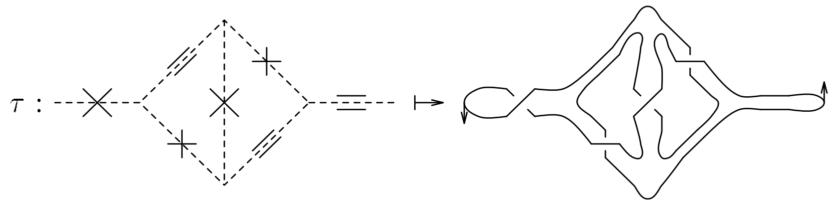
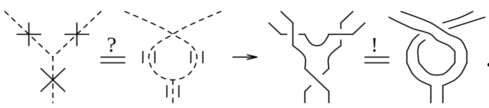


FIGURE 15. A marked diagram and its thickening. In this case, the resulting marked surface is a torus with one boundary component, marked by two arrows of the same orientation.

Exercise 6.7. Show that the definition of τ *does not* depend on the planar projection of C. Recall that each of the vertices of a marked diagram C is oriented. Check that τ can be defined using this information only, and that (if the correct choices are made) the resulting definition agrees with that of figure 15 if a diagram is planar and its vertices are oriented counterclockwise.

Proposition 6.8. *The map τ descends to a map $\tau : \mathcal{B}^m \to \mathcal{M}$.*

Proof. Let us show that when two marked diagrams in the AS identity are thickened, the resulting marked surfaces are isomorphic:



.

A similar check shows the same for the two sides of the IH identity. $\square$

Theorem 14. *The map τ is an isomorphism.*

Proof. Very briefly, marked diagrams with no univalent vertices correspond to cells of maximal dimension in a certain triangulation (see *e.g.* Penner [32]) of the moduli space of Riemann surfaces, and our theorem can be deduced from the fact that Moduli space is connected. A direct combinatorial proof of theorem 14 will appear in Bar-Natan, [9]. $\square$

Definition 6.9. Define the map $\Phi : \mathcal{B} \to \mathcal{M}$ to be the composition $\Phi = \tau \circ \mu$.

For the purpose of conjecture 2, much of the information in $\mathcal{M}$ is superfluous:

Exercise 6.10. For a marked surface $M \in \mathcal{M}$ define a linear functional $\check{M} \in \mathcal{M}^*$ by

$$\check{M}(M') = \begin{cases} 1 & M' \simeq M, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M_\uparrow$ and $M_\downarrow$ be two marked surfaces that differ only by the orientation of one of their marks. Then the corresponding linear functionals on $\mathcal{B}$ differ only by a sign:

$$\Phi^* \check{M}_\uparrow + \Phi^* \check{M}_\downarrow = 0.$$

Definition 6.11. Each tangent to the boundary of a connected surface M defines an orientation on the boundary component on which it lives, and if M is orientable, such a tangent taken together with the outward pointing normal defines an orientation on M. Call a connected marked surface M *normalized* if either of the following holds:

- M is orientable and all its markings generate the same orientation on M.
- M is non-orientable and its marking generate *consistent* orientations on each boundary component separately.

A general marked surface is called *normalized* if all of its connected components are normalized. It is called *normalized orientable* if all of its connected components are normalized and orientable.

Remark 6.12. Exercise 6.10 shows that in order to study $\Phi^* \mathcal{M}^*$ it is sufficient to consider only the linear functionals corresponding to normalized marked surfaces.

Exercise 6.13. Let M be a connected marked surface of genus g. If M has b boundary components and n tangents marked on its boundary define the degree m of M by

$$m = \begin{cases} 2g + b + n - 2 & \text{if } M \text{ is orientable,} \\ g + b + n - 1 & \text{otherwise.} \end{cases}$$

Define the degree of a general marked surface to be the sum of the degrees of its connected components, and check that with this grading on $\mathcal{M}$, the map Φ preserves degrees.

Remark 6.14. The definition of the thickening map has a natural extension (also denoted by τ) to marked diagrams which are allowed to have some p cycles of directed full lines. Simply thicken dashed lines as before, and thicken directed lines to semi-open strips, so that every full line l is replaced by $l \times [0, 1)$. The result is a *punctured* marked surface — a marked surface with p points removed. See figure 16.

Exercise 6.15. Check that the extended map τ satisfies the STU relation and thus descends to a map defined on the relevant quotient space.

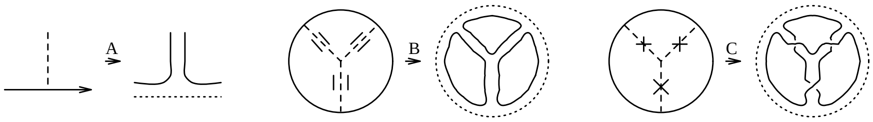


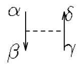
FIGURE 16. Thickening full lines. A shows how to thicken dashed-full vertices, in B the resulting surface is a punctured sphere with three holes while in C it is a punctured torus. In all cases a dotted line represents an 'open end' of the surface.

6.3. The classical Lie algebras. The purpose of this section is to show how to compute $\mathcal{T}(D)$ for an arbitrary diagram $D \in \mathcal{D}^t$ and a large number of representations of the algebras in the families gl, su, sp, and so. I will show in detail the computations for so, and just state the results for gl, su, and sp.

6.3.1. *Chord diagrams in the N dimensional representation of $so(N)$.* For a start, let us consider only chord diagrams. Let Λ^1 be the defining representation of $so(N)$ for some N, and let t be given by matrix trace in that representation[12],

$$t(a,b) = \operatorname{tr}\Lambda^1(a)\Lambda^1(b).$$

The basic building block of chord diagrams is the tensor

$$\mathcal{T}^{\alpha\gamma}_{\beta\delta} = r^{\alpha}_{b\beta}(t^{-1})^{ab} r^{\gamma}_{a\delta} =$$  $$\in ((\Lambda^1)^*)^{\otimes 2} \otimes (\Lambda^1)^{\otimes 2},$$

where α and γ run over a basis of $(\Lambda^1)^*$, β and δ run over a basis of Λ^1, a and b run over a basis of $so(N)$, and $r^{\alpha}_{b\beta}$ and $(t^{-1})^{ab}$ are the tensors considered in section 2.4, expressed in coordinates.

A convenient choice of generators for $so(N)$ are the $N \times N$ matrices M_{ij} $(i<j)$, given by

$$(M_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}.$$

That is, the ij entry of M_{ij} is $+1$, the ji entry of M_{ij} is -1, and all other entries of M_{ij} are zero. The invariant bilinear form that we pick on $so(N)$ is the matrix trace in the defining representation, and so

$$t_{(ij)(kl)} \overset{\text{def}}{=} \operatorname{tr}(M_{ij}M_{kl}) = -2\delta_{ik}\delta_{jl}.$$

Inverting the $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ matrix $t_{(ij)(kl)}$ we get

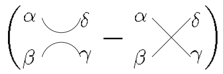
$$(t^{-1})^{(ij)(kl)} = -\frac{1}{2}\delta^{ik}\delta^{jl}, \tag{27}$$

[12]In a simple Lie algebra, an invariant metric is always a multiple of the Killing form. It is easy to check that if $t \to \kappa t$, then $\mathcal{T}(D) \to \kappa^{-\deg D}\mathcal{T}(D)$, and so our choice of t is as good as any other choice.

and so

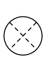
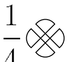
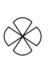
$$\mathcal{T}^{\alpha\gamma}_{\beta\delta} = \sum_{i<j;k<l} (t^{-1})^{(ij)(kl)} (M_{ij})_{\alpha\beta}(M_{ij})_{\gamma\delta}. \tag{28}$$

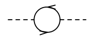
Using (27) and some algebraic manipulations we can simplify (28), and then represent it by a diagram:

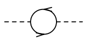
(29) $$(28) = \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) = \frac{1}{2}$$ .

The last thing to note is that $\mathcal{T}(k \text{ disjoint circles}) = N^k$. These rules are sufficient to compute $\mathcal{T}(D)$ for any chord diagram D.

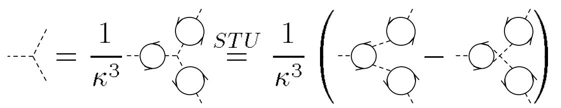
Example 6.16.

 $\to \frac{1}{4}$  $- \frac{1}{2}$  $+ \frac{1}{4}$  $\to \frac{N}{4} - \frac{N}{2} + \frac{N^2}{4} = \frac{N(N-1)}{4}.$

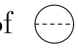
6.3.2. *Diagrams with trivalent vertices.* By proposition 2.10 the tensor  is invariant, and therefore if the Lie algebra $\mathfrak{g}$ is simple, it must be a multiple of $t =$ ----. In that case, let κ be the constant for which

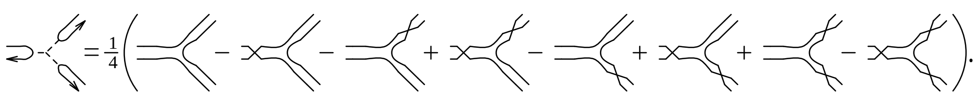
(30)  $= \kappa$----.

Using this definition, diagrams that have trivalent vertices can be reduced to diagrams without trivalent vertices:

(31) 

(The above equality should be read as an equality between the tensors represented by these diagrams.)

In the case of the N dimensional representation of $so(N)$, a comparison of  and  (using (29)) shows that $\kappa = 1$. Furthermore, in this case (31) can be further reduced using (29). The result is:

(32) 

In other words for $D \in \mathcal{D}^t$,

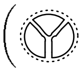
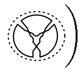
$$\mathcal{T}(D) = 2^{v-e} \sum_M s_M N^{b(\tau D_M)}, \tag{33}$$

where

- v (e) is the number of internal vertices (edges) in D.
- The sum is over all possible markings M of the dashed lines of D, as in definition 6.4.

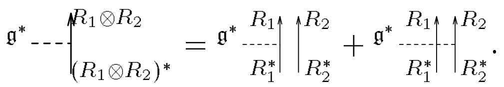
- s_M is the sign corresponding to M as in definition 6.4.
- $b(\tau D_M)$ is the number of boundary components of the thickening $\tau(D_M)$ of τD_M, defined in remark 6.14.

Example 6.17. $\mathcal{T}\Big($  $\Big)$ can be computed in the following manner:

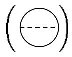
$2^{1-3}\Big($  $-\,3$  $+\,3$  $-$  $\Big) \to 2^{-2}\left(N^3-3N^2+3N-N\right) = \dfrac{N(N-1)(N-2)}{4}.$

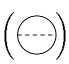
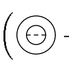
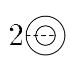
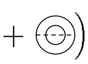
6.3.3. *Tensor products of representations.* Recall that as in remark 2.12 we can consider diagrams in which each directed arc is colored by a (possibly) different representation. The following proposition can be used to reduce computations with arcs colored by a tensor product $R_1 \otimes R_2$ to computations with arcs colored by R_1 or R_2:

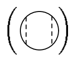
Proposition 6.18. *Let R_1 and R_2 be two representations of some Lie algebra $\mathfrak{g}$. The following is an equality between tensors in $\mathfrak{g}^* \otimes R_1 \otimes R_2 \otimes R_1^* \otimes R_2^*$:*



Proof. Follows from the definition of the action of $\mathfrak{g}$ on a tensor product, $(R_1 \otimes R_2)(\mathfrak{g}_a) = R_1(\mathfrak{g}_a) \otimes 1 + 1 \otimes R_2(\mathfrak{g}_a)$. $\square$

Example 6.19. Let us compute $\mathcal{T}\Big($  $\Big)$ in the second tensor power of the defining representation of $so(N)$. In the formula below, full lines are colored by the *defining* representation.

$\mathcal{T}\Big($  $\Big) = \mathcal{T}\Big($  $+\,2$  $+$  $\Big) = \dfrac{1}{2}N^2(N-1) + 0 + \dfrac{1}{2}N^2(N-1) = N^2(N-1).$

Exercise 6.20. Verify that $\mathcal{T}_{(\Lambda^1)^{\otimes 2}}\Big($  $\Big) = N(N-1)(N^2-N+2)$.

Definition 6.21. Let $K : (S^1 = \{z : |z| = 1\}) \to \mathbf{R}^3$ be a framed knot and let $n(z)$ $(z \in S^1)$ be a section of the normal bundle of K compatible with its framing. For a non-zero integer q define the qth *disconnected cabling* $K^{\otimes q}$ of K to be the q-component link whose jth component $K_j^{\otimes q}$ is given by

$$K_j^{\otimes q}(z) = K(z + j\epsilon n(z)),$$

where ϵ is a very small number. Let $K^{\otimes q}$ inherit the framing of K.

Exercise 6.22. Show that if V is a Vassiliev invariant of type m of framed knots then $V \circ (K \to K^{\otimes q})$ is also a Vassiliev invariant of type m. Show that the operations $K \to K^{\otimes q}$ on knots and $R \to R^{\otimes q}$ on representations are adjoints of each other — show that if V is a Vassiliev invariant of type m of framed knots for which $W_m(V) = W_{\mathfrak{g},R,m}$, then $W_m(V \circ (K \to K^{\otimes q})) = W_{\mathfrak{g},R^{\otimes q},m}$.

6.3.4. *The Adams operations.* Let q be a non-negative integer. If χ is a virtual character on a compact Lie group G (*i.e.*, a conjugation invariant L^2 function on G), then so is $\bar{\psi}^q\chi$, which is defined by $(\bar{\psi}^q\chi)(g) = \chi(g^q)$. This defines an operation $\bar{\psi}^q$ called *the q'th Adams operation* on the representation ring $R(G)$ of G (see *e.g.* [11, pp. 104]). We are interested in this operation on the Lie algebra level. Recall that every character (and hence every virtual character χ) induces a linear functional $\phi(\chi)$ on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G.

Exercise 6.23. Let m denote the product of and let $\Delta_{\mathfrak{U}}$ denote the co-product (see *e.g.* [31]) of $\mathfrak{U}(\mathfrak{g})$, and let $\hat{\psi}^q : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ be the composition

$$\mathfrak{U}(\mathfrak{g}) \xrightarrow{\Delta_{\mathfrak{U}}} \mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g}) \xrightarrow{1\otimes\Delta_{\mathfrak{U}}} \dots \xrightarrow{1^{\otimes(q-2)}\otimes\Delta_{\mathfrak{U}}} \mathfrak{U}(\mathfrak{g})^{\otimes q} \xrightarrow{m} \mathfrak{U}(\mathfrak{g}).$$

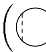
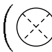
Then the operations $\bar{\psi}^q$ and $\hat{\psi}^q$ correspond under ϕ. In other words,

$$\phi(\bar{\psi}^q\chi) = (\hat{\psi}^q)^*\phi(\chi).$$

Exercise 6.24. Fix a Lie algebra $\mathfrak{g}$ and a metric t, let D be a diagram with a distinguished cycle of directed lines, let $\mathcal{T}_\chi(D)$ denote the tensor associated with the diagram D when its distinguished cycle is colored with the virtual representation χ, and recall that $\psi^k D$ was defined in definition 3.11. Show that

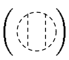
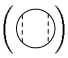
$$\mathcal{T}_{(\hat{\psi}^q)^*\chi}(D) = \mathcal{T}_\chi(\psi^q D).$$

Example 6.25. For $\mathfrak{g} = so(N)$ and Λ^1 the defining representation,

$\mathcal{T}_{(\hat{\psi}^2)^*\Lambda^1}$ () $= \mathcal{T}_{\Lambda^1}$ (ψ^2 ) $= 12\mathcal{T}_{\Lambda^1}$ () $+ 4\mathcal{T}_{\Lambda^1}$ () $= N(N-1)(3N-2).$

In [2] it is shown that in an arbitrary *special* λ-ring the operations ψ^q are given by universal polynomials in the 'exterior power' operations Λ^q. In fact, the opposite is also true and the Λ^q's can also be expressed in terms of the ψ^q's. We have simple combinatorial algorithms for computations in the defining representation of $so(N)$, for taking tensor products and for applying the operations ψ^q. Combining all of these we see that we have effective algorithms for computations in an arbitrary representation belonging to the λ-ring spanned by the defining representation.

Example 6.26. Using the symbol $\mathfrak{g}$ also for the adjoint representation of $\mathfrak{g} = so(N)$, we have $\mathfrak{g} = \Lambda^2 = \frac{1}{2}\left((\Lambda^1)^{\otimes 2} - (\hat{\psi}^2)^*\Lambda^1\right)$ and thus

$\mathcal{T}$ () $= \mathcal{T}_{\mathfrak{g}}$ () $= \dfrac{N(N-1)(N^2-N+2)-N(N-1)(3N-2)}{2} = \dfrac{N(N-1)(N-2)^2}{2}.$

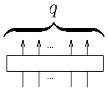
Exercise 6.27. Use (33) to verify the above result.

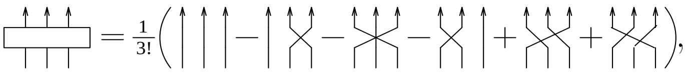
Exercise 6.28. Let $\Lambda^N = \Lambda^N_+ \oplus \Lambda^N_-$ be the decomposition into irreducible representations of the representation Λ^N of $so(2N)$. Show that for a diagram $D \in \mathcal{D}^t$ one has $\mathcal{T}_{\Lambda^N_+}(D) = \mathcal{T}_{\Lambda^N_-}(D)$.

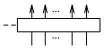
Hint 6.29. Use the fact that $so(2N)$ has an outer automorphism that interchanges Λ^N_+ and Λ^N_-.

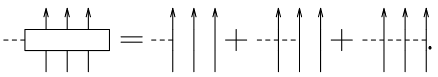
Remark 6.30. Exercise 6.28 implies that $\mathcal{T}_{\Lambda^N_\pm}(D) = \frac{1}{2}\mathcal{T}_{\Lambda^N}(D)$, and thus we have an effective algorithm for computing $\mathcal{T}_{\Lambda^N_\pm}(D)$. Notice that the representations Λ^0, Λ^1, $\dots$, Λ^{N-1}, $\Lambda^N_\pm$ span the representation ring of the *group*[13] $SO(2N)$ and that the representations Λ^0, Λ^1, $\dots$, Λ^N span the representation ring of the group $SO(2N+1)$. We see that the results of this section allow us to compute $\mathcal{T}_R(D)$ for *any* representation R of the group $SO(N)$.

6.3.5. *Exterior powers.* The previous section had already given us (indirect) means to compute $\mathcal{T}_{\Lambda^q R}$ for representations R for which we know how to compute $\mathcal{T}_R$. The following exercise gives a more direct way to do the same:

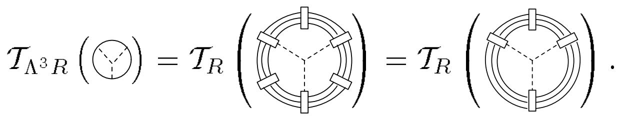
Exercise 6.31. Let  denote the qth total antisymmetrization tensor — for example,



Let  denote the standard action of $\mathfrak{g}$ on $R^{\otimes q}$ as in proposition 6.18:



Prove the obvious generalization of the following statement:



6.3.6. *Other Lie algebras.* So far we have dealt only with a single Lie algebra, and varied its representation. Let us now try to vary the algebra:

Exercise 6.32. Let $(\mathfrak{g}_{\mathbf{C}}, R_{\mathbf{C}})$ be the complexification of a pair $(\mathfrak{g}, R)$ consisting of a real Lie algebra $\mathfrak{g}$ and a representation R of $\mathfrak{g}$ in a real vector space. Show that $W_{\mathfrak{g}_{\mathbf{C}},R_{\mathbf{C}},m} = W_{\mathfrak{g},R,m}$ for every m (and therefore no new information can be gained by studying, say, the various real forms of $so(N,\mathbf{C})$).

Exercise 6.33. External tensor products correspond to the co-product on $\mathcal{A}$ (or equivalently, to the product in $\mathcal{A}^*$). In other words, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras, let t_1 and t_2 be metrics and let R_1 and R_2 be representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.

[13] Notice the distinction between the group $SO(N)$ and the Lie algebra $so(N)$. The Lie algebra has spin representations that do not integrate to single-valued representations of the group.

Consider the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the metric $t_1 \oplus t_2$ and the external tensor product representation $R_1 \otimes R_2$. Then the following diagram is commutative:

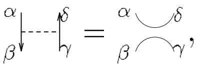
$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\mathcal{T}_{R_1 \otimes R_2}"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mathcal{T}_{R_1} \otimes \mathcal{T}_{R_2}"] \\ \mathbf{C} & \mathbf{C} \otimes \mathbf{C} \arrow[l, "m"'] \end{tikzcd}$$

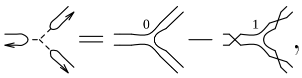
Exercise 6.34. Show that a linear functional ω on $\mathcal{A}^*$ associated to some one dimensional representation of a one dimensional Abelian Lie algebra is an (easily describable) unit in the algebra $\mathcal{A}^*$. Show that the same linear functionals on $\mathcal{G}_m \mathcal{A}^*$ appear as $W_m(V)$ where V is the coefficient of x^m in a framed-knot invariant of the form $K \to \exp(rxw(K))$, r is an arbitrary constant, and $w(K)$ is the *writhe* (see *e.g.* [22]) of K. Show that $\omega \in (\Phi \circ \bar{\sigma})^* \mathcal{M}^*$.

Exercise 6.35. Check that higher dimensional Abelian Lie algebras give the same linear functionals on $\mathcal{A}$ as one dimensional Lie algebras.

Exercises 6.33 and 6.34 taken together show that (in our context) studying the sl family is essentially equivalent to studying the gl family. So let us pick one of them:

Exercise 6.36. The $gl(N)$ analogs of equations (29), (32) and (33) are

(34) $$\mathcal{T}^{\alpha\gamma}_{\beta\delta} =$$ 

(35) 

and

(36) $$\mathcal{T}(D) = \sum_M (-1)^{s_M} N^{b(\tau D_M)},$$

where here

- The sum is over all possible markings M of the internal vertices of D by the digits 0 and 1.
- s_M is the sum of all the digits in M.
- $b(\tau D_M)$ is the number of boundary components of the thickening $\tau(D_M)$ of τD_M. Internal vertices are thickened depending on the digit marked on them using one of the two possibilities in (35) and chords are always thickened as if they were marked by an '=' symbol.

Notice that the exterior powers of the defining representation of $gl(N)$ generate its representation ring, and thus we have effective computation techniques for $gl(N)$ in all of its representations.

Exercise 6.37. Find the analogs of equations (29), (32) and (33) for the Lie algebra $sp(N)$ or read them in Bar-Natan, [5, 6]. Use straightforward (but rather tricky) combinatorics to show that if D is a diagram with no univalent vertices but with an arbitrary number of cycles of directed lines marked by the defining representation, then

$$\mathcal{T}_{sp(N)}(D) = \mathcal{T}_{so(-2N)}(D) \tag{37}$$

where $\mathcal{T}_{so(-2N)}$ is defined as in (33), only with $(-2N)$ replacing N.

Problem 6.38. I learned about equation (37) from M. Kontsevich [25]. He claims that the above result follows from studying the super Lie algebra $Osp(m,n)$. I believe, but as of now I can not reproduce his result.

Notice that the exterior powers of the defining representation of $sp(N)$ generate its representation ring, and thus we have effective computation techniques for $sp(N)$ in all of its representations.

Problem 6.39. The results of Kuperberg [27] can be used to derive an explicit algorithm for computations using the rank 2 exceptional Lie algebra G_2 (see also [35]). I do not know whether the results are inside $(\Phi \circ \bar{\sigma})^* \mathcal{M}^*$.

6.4. Marked surfaces and the classical Lie algebras. In this section we will prove theorem 11 and a few related results. We will do so by constructing a vector space $\mathcal{L}$ and a pairing $\langle \cdot, \cdot \rangle : \mathcal{M} \otimes \mathcal{L} \to \mathbf{F}$ that has the following properties:

Proposition 6.40. *(Proof on page 52) For an element $L \in \mathcal{L}$ denote by $\Psi(L)$ the pullback to $\mathcal{B}^*$ via the map Φ of the linear functional $(M \mapsto \langle M, L \rangle) \in \mathcal{M}^*$. Then for every list of triples $\{(\mathfrak{g}_i, R_i, m_i)\}$ where each $\mathfrak{g}_i$ is either an so or a gl algebra with R a single-valued representation of the corresponding group there is a (non-unique) generator $L \in \mathcal{L}$ for which*

$$\Psi(L) = \bar{\chi}^* \prod_i W_{\mathfrak{g}_i, R_i, m_i}. \tag{38}$$

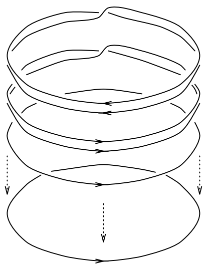
Conversely, for every generator $L \in \mathcal{L}$ there is a (canonically determined) list of triples $\{(\mathfrak{g}_i, R_i, m_i)\}$ for which (38) holds. (For the definition of the isomorphism $\bar{\chi}$ see page 34).

Proposition 6.41. *(Follows from (37) and exercises 6.34 and 6.51) Let m be a non-negative integer. If $\mathfrak{g}$ is a symplectic or an Abelian algebra and R is a representation of $\mathfrak{g}$ or if $\mathfrak{g}$ is in the family so and R is a spin representation, then $W_{\mathfrak{g},R,m} \in (\Phi \circ \bar{\sigma})^* \mathcal{M}^*$.*

Proposition 6.42. *(Proof on page 53) The pairing $\langle \cdot, \cdot \rangle$ is non-degenerate in the sense that if $\bar{M}_1$ and $\bar{M}_2$ are linear combinations of normalized marked surfaces and $\langle \bar{M}_1, L \rangle = \langle \bar{M}_2, L \rangle$ for every $L \in \mathcal{L}$, then $\bar{M}_1 = \bar{M}_2$.*

Clearly, these three propositions together with remark 6.12 imply theorem 11.

6.4.1. *The pairing* $\langle\cdot,\cdot\rangle$. By a *covering* of the circle we will mean a not necessarily connected oriented covering of the oriented circle, regarded up to an orientation preserving homeomorphism. Such coverings can be identified with non-empty finite sets[14] of integers measuring the *multi-degree* of a cover:



$$\longleftrightarrow \qquad \{-2,2,1\}.$$

Exercise 6.43. Explain how coverings of the circle correspond to general cablings of framed knots, generalizing definitions 3.13 and 6.21. Generalize exercises 3.14 and 6.22 to general cablings.

A *labeled covering* will be a quadruple $L=(s,N,L_0,m)$ where L_0 is a covering of the circle, m is a non-negative integer called 'the degree of L', s is called 'the symbol of L' and is either the symbol 'gl' or the symbol 'so', and N is an integer. A *covering bouquet* is a finite non-empty set of labeled coverings of the circle — something that looks like

$$\{(gl,54,\{2,2,1\},6)\,,(so,9,\{7,3\},14)\,,\dots\}\,.$$

Definition 6.44. Let $\mathcal{L}$ be the vector space spanned by the collection of all covering bouquets.

Definition 6.45. Let $L=(s,N,L_0=\{q_1,\dots,q_k\},m)$ be a labeled covering of the circle and let M be a marked surface with n marked tangents. Define

$$\langle M,L\rangle=\begin{cases}0 & \text{if } \deg M\neq m,\\ 0 & \text{if } s=\text{`}gl\text{'} \text{ and } M \text{ is not normalized orientable,}\\ \displaystyle\sum_{\pi\in\mathcal{S}_n}\ \sum_{\substack{\text{all liftings}\\ l \text{ of } \pi}} N^{b(\tau L_0\cup_l M)} & \text{otherwise,}\end{cases}\tag{39}$$

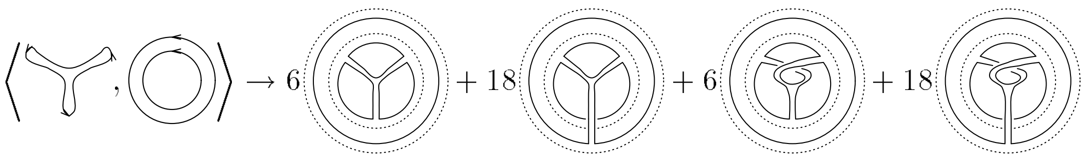
where:

- The outer sum is over the $n!$ ways (each indexed by a permutation $\pi\in\mathcal{S}_n$) of arranging the n markings of M around the base S^1 of the covering L_0.
- The inner sum is over the $(\sum|q_i|)^n$ ways of lifting π from the base S^1 to the covering itself.

[14] In this section the word "set" means 'an unordered collection with multiplicities'. I.e., $\{2,2,1\}=\{2,1,2\}=\{1,2^2\}\neq\{1,2\}$. Notice that for a set $\{q,\dots,q\}$ of size β we sometimes use the notation $\{q^\beta\}$.

- τL_0 is the thickening of L_0 into a disjoint union of k punctured disks as described in exercise 6.14.
- $\tau L_0 \cup_l M$ is the punctured surface (with boundary) obtained by gluing M to τL_0 in the n sites specified by l, in such a way that in each such site the orientation given on L_0 matches with the corresponding marking of M.
- $b(\tau L_0 \cup_l M)$ is the number of boundary components of $\tau L_0 \cup_l M$.

Example 6.46. Let us compute the pairing of a disk marked by three aligned tangents with the labeled covering $(gl, N, \{1,1\}, 2)$. The sum is over 48 terms, but after dividing by the 3-fold cyclic symmetry of the circle and by the possibility to exchange the two leafs of the covering, we get only 4 types of contributions. Observed from above, this is how it looks:



$$\to 6N^4 + 18N^2 + 6N^2 + 18N^2 = 6N^2(N^2+7).$$

Definition 6.47. Let $\{M_i\}_{i=1}^c$ be the connected components of a general marked surface $M \in \mathcal{M}$ and $L = \{L_k\}_{k=1}^{\gamma}$ be a presentation of a covering bouquet as a set of labeled coverings. Define

$$\langle M, L\rangle = \sum_{k_1,\dots,k_c=1}^{\gamma} \prod_{k=1}^{\gamma} \left\langle \bigcup_{\{i:k_i=k\}} M_i, L_k \right\rangle,$$

and extend $\langle\cdot,\cdot\rangle$ linearly to all of $\mathcal{M}\otimes\mathcal{L}$.

6.4.2. *Proof of proposition* 6.40.

Lemma 6.48. *For any m and N, $\Psi(\{(so, N, \{1\}, m)\}) = W_{so(N),\Lambda^1,m}$.*

Proof. Follows immediately from the definition of the pairing $\langle\cdot,\cdot\rangle$, from the definition (6.9) of the map Φ, from the definition of $\bar\chi$, and from (33). □

Lemma 6.49. *For any m and N, $\Psi(\{(gl, N, \{1\}, m)\}) = W_{gl(N),\Lambda^1,m}$.*

Proof. Consider a Chinese character C as a one dimensional CW-complex. A marking $\tilde C$ of C defines a co-chain $\beta(\tilde C) \in C^1(C, \mathbf{Z}/2\mathbf{Z})$ — for an edge l in C define

$$\beta(\tilde C)(l) = \begin{cases} 0 & l \text{ is marked by an `$=$' symbol,} \\ 1 & l \text{ is marked by a `$\times$' symbol.} \end{cases}$$

It is easy to check that the thickening $\tau\tilde C$ of $\tilde C$ is orientable iff $\beta(\tilde C)$ is a co-boundary, *i.e.* iff there exists an $\alpha \in C^0(C, \mathbf{Z}/2\mathbf{Z})$ for which $\beta(\tilde C) = d\alpha$. Also, $\tau\tilde C$ is normalized orientable iff such an α can be found which vanishes on the external (univalent) vertices of C. In such a case, α is uniquely determined by $\beta(\tilde{C})$. In computing $\Psi(\{(gl,N,\{1\},m)\})$ a signed summation over normalized orientable thickenings of C is performed (see definition 6.4 and 6.45), and by the above discussion it can be replaced by a signed summation over $\alpha \in C^0(\text{int } C, \mathbf{Z}/2\mathbf{Z})$. Comparing with (36), we see that we are done. $\square$

Exercise 6.50. Let Λ^1 be the defining representation of $so(N)$ or $gl(N)$, and let P^q denote the virtual representation $(\hat{\psi}^q)^*\Lambda^1$. Use proposition 6.18, exercises 6.23 and 6.33 and equation (26) to prove the obvious generalization of the statement:

$$\Psi(\{(gl,54,\{2,2,1\},6),(so,9,\{7,3\},14)\}) = \bar{\chi}^*\left(W_{gl(54),P^2\otimes P^2\otimes P^1,6}\cdot W_{so(9),P^7\otimes P^3,14}\right).$$

Clearly, lemmas 6.48 and 6.49 and exercise 6.50 prove proposition 6.40. The following exercise is the remaining ingredient for the proof of proposition 6.41:

Exercise 6.51. The crux of the argument leading to computations for the Lie algebra so is equation (32), used to simplify internal vertices of a diagram. This equation is valid no matter what representation is used on the external circle surrounding a diagram in $\mathcal{D}^t$. Use this fact to show that if R is a spin representation of $so(N)$ (or, in fact, any representation of $so(N)$), then $W_{so(N),R,m} \in (\Phi\circ\bar{\sigma})^*\mathcal{M}^*$.

6.4.3. *Proof of proposition* 6.42. The idea is to show that all relevant information about a normalized marked surface M can be read from the numbers $\langle M,L\rangle$ for various L's. Keeping exercise 6.13 and the classification of 2D surfaces (see *e.g.* [30]) in mind we see that (for a connected normalized surface) it is enough to read its degree, whether it is orientable, and the number of markings on each of its boundary components.

Exercise 6.52. Let M be a disk with n aligned tangents marked on its boundary (so that $\deg M = n-1$). Prove that the highest power of N in $\langle M,(so \text{ or } gl,N,\{1\},n-1)\rangle$ is N^n and that in the sum (39) this power is achieved only when the lifting l arranges the markings of M along the leaf S^1 of the covering $\{1\}$ *in order*. Conclude that

$$\langle M,(so \text{ or } gl,N,\{1\},n-1)\rangle = nN^n + (\text{lower order terms}).$$

Exercise 6.53. The same is still true for the cover $\{q\}$. Only that in this case there are more liftings of the markings of M to the leaf of the cover that keep them in order. More precisely, let p be the number of non-decreasing sequences of length $n-1$ of non-negative integers smaller than q. Prove that

$$\langle M,(so \text{ or } gl,N,\{q\},n-1)\rangle = qpN^n+\ldots = q\binom{q+n-2}{n-1}N^n+\ldots$$

$$= \left(\frac{q^n}{(n-1)!} + (\text{lower powers of } q)\right)N^n+\ldots.$$

Exercise 6.54. Let $M = \bigcup_1^b M_i$ be a disjoint union of b disks with n_i aligned tangents marked on the boundary of the ith disk, $1 \leq i \leq b$. Let m be the degree of M, let $n = \sum n_i$ be the total number of markings on M and let $L^\beta(N)$ be the labeled covering $\left(so \text{ or } gl, N, \{1^\beta\}, m\right)$. Prove that the highest power of N in $\langle M, L^\beta(N)\rangle$ is $N^{\beta-b+n}$, and it is attained in the sum (39) only in the cases were the lifting l satisfies:

- For each i, all the markings on M_i lift to the same component of L^β, and the cyclic ordering induced on them from the cyclic ordering the points of that component agrees with their cyclic order on the boundary of M_i.
- If, for some i and j, the disks M_i and M_j lift to the same component L_0 of L^β, then they are *non-interlaced* — L_0 can be cut into two disjoint arcs A_i and A_j, so that A_i (A_j) is connected only to M_i (M_j).

Definition 6.55. The *width* of a monomial Q in the variables $q_1, \dots q_\beta$ is the number of the q_i's that appear in Q in a positive power. For example, the width of the monomial x^2y^3z is 3. The *maximal width* of a polynomial P is the maximal width of a monomial in P.

Exercise 6.56. Prove that if M is as in the previous exercise and $L^\beta(N)$ is the labeled covering $(so \text{ or } gl, N, \{q_1, \dots, q_\beta\}, m)$, then the same conclusion as in the previous exercise holds. Let Q be the coefficient of $N^{\beta-b+n}$ in $\langle M, L^\beta(N)\rangle$. Prove that Q is a polynomial of maximal width $\min(b, \beta)$ in $q_1, \dots q_\beta$, and that if in addition it is given that $\beta \geq b$ then

$$Q = \left(b! \sum_{1\leq j_1<\dots<j_b\leq\beta} \prod_{i=1}^{b} \frac{q_{j_i}^{n_i}}{(n_i-1)!}\right) + \left(\begin{array}{c}\text{terms of lower width or}\\ \text{order in } q_1, \dots q_\beta\end{array}\right). \tag{40}$$

Notice that the above exercise shows that if M is a connected normalized surface than its isomorphism class can be read from the numbers $\langle M, L\rangle$ for various L. The degree of M can be read from the degree of any L for which $\langle M, L\rangle \neq 0$. M is *not* orientable iff $\langle M, L\rangle = 0$ for all L of symbol gl, b can be read by increasing β until the width of Q stops growing, and then the number of markings on each component of the boundary of M can be read from the powers of the q_j's appearing in (40).

If M is a general normalized surface, we will show that its isomorphism class can be read from the numbers $\langle M, L\rangle$ where L is a bouquet $L = \cup L_k^{\beta_k}(N_k)$ where each $L_k^{\beta_k}(N_k)$ is a labeled covering of the type considered in exercise 6.56:

$$L = \left\{(s_1, N_1, \{q_{11}, \dots, q_{1\beta_1}\}, m_1), \dots, (s_\gamma, N_\gamma, \{q_{\gamma 1}, \dots, q_{\gamma\beta_\gamma}\}, m_\gamma)\right\},$$

here each s_k is either so or gl.

Exercise 6.57. For fixed s_k's and fixed m_k's, show that $P = \langle M, L\rangle$ is a polynomial in the variables $N_1, \dots, N_\gamma, q_{11}, \dots, q_{\gamma\beta_\gamma}$ and that this polynomial is divisible by $\prod_{k=1}^{\gamma} N_k$.

We will now define an ordering $\preceq$ among the monomials that appear in $\langle M, L\rangle$ and show that from the $\preceq$-maximal monomial in $\langle M, L\rangle$ the isomorphism class of M can be read.

Definition 6.58. Order monomials in the variables $N_1, \dots, N_\gamma, q_{11}, \dots, q_{\gamma\beta_\gamma}$ which are divisible by $\prod N_k$ lexicographically according to the following parameters of such a monomial Q:

- The total degree of Q in the variables $N_1, \dots, N_\gamma$.
- The width of $Q\,/(\prod N_k)$ in the variables $N_1, \dots, N_\gamma$.
- The width of Q in the variables $q_{11}, \dots, q_{\gamma\beta_\gamma}$.
- The power of q_{11} in Q.
- The power of q_{12} in Q.

$$\vdots$$

- The power of $q_{\gamma\beta_\gamma}$ in Q.

Exercise 6.59. Let M be a normalized marked surface. For a fixed γ, set $s_k = so$, $(1 \le k \le \gamma)$. Find some values for $m_1, \dots, m_\gamma$ for which $\langle M, L\rangle$ is non-zero (show that this is possible whenever γ is sufficiently large), and call the resulting polynomial P_γ. Let c be the maximal value of γ for which the maximal width w_γ of $P_\gamma\,/(\prod N_k)$ is still equal to γ (show that such a value of γ exists and that w_γ is constant for $\gamma \ge c$). For that value of γ, fix $\beta_1, \dots, \beta_\gamma$ for which the maximal width of P_γ in the variables $q_{11}, \dots, q_{\gamma\beta_\gamma}$ has reached its maximal possible value (show that this 'settling of width' indeed happens). Let Q be a $\preceq$-maximal monomial in P_γ. Show that M is the union of c connected components (orientable or not) $M_1, \dots, M_c$ satisfying:

- $\deg M_k = m_k$.
- The width b_k of Q in the variables $q_{k1}, \dots, q_{k\beta_k}$ is equal to the number of boundary components of M_k.
- Order the boundary components of M_k in some way $B_{k1}, \dots, B_{kb_k}$ so that the number n_{kj} of markings on B_{kj} is a non-increasing function of j. Then n_{kj} is equal to the degree of q_{kj} in Q.

Exercise 6.60. For a fixed γ and $m_1, \dots, m_\gamma$ as before, find a maximal set S of s_k's that can be changed from so to gl so that the $\preceq$-order of the resulting Q remains the same as in the previous exercise (such a set might be empty). Show that the components M_k corresponding to the s_k's in S are orientable.

The last two exercises show that the isomorphism class of M can be read from the numbers $\langle M, L\rangle$. We now just need to show that if $\bar{M}$ is a linear combination of normalized marked surfaces and $\langle \bar{M}, L\rangle = 0$ for all L's, then $\bar{M} = 0$.

Exercise 6.61. Show that this is indeed the case.

Hint 6.62. Define an ordering relation $\ll$ on normalized marked surfaces in a way similar to the definition of $\preceq$. Show that the coefficient of the $\ll$-maximal surface appearing in $\bar{M}$ with a non-zero coefficient can be read from $\langle \bar{M}, L\rangle$ using the previous two exercises. $\square$

6.5. Marked surfaces and the classical knot polynomials. Using everything that we already know, the simplest way to prove theorem 12 is to recall (see Turaev [35, 37, 41]) that the HOMFLY polynomial is built from the Lie algebra $sl(N)$ (or from the essentially equivalent (see exercise 6.34) Lie algebra $gl(N)$), that the Kauffman polynomial is built from the Lie algebra $so(N)$, and that cabling these polynomials corresponds to taking higher representations of those groups. Remembering all that, theorem 12 follows immediately from theorem 11. Alternatively, if one does not want to use theorem 5, it is easy to get equations (34) and (36) from the power series expansion in the variable x of a framing dependent version of the HOMFLY polynomial,

$$P^f\left(\nwarrow\!\!\!\!\nearrow\right) - P^f\left(\nwarrow\!\!\!\!\nearrow\right) = (e^{x/2} - e^{-x/2})P^f\left(\,\rangle\,\langle\,\right);$$
$$P^f\left(\uparrow\!\circlearrowright\right) = e^{Nx/2}P^f\left(\,\uparrow\,\right),$$

and to get equations (29) and (33) from the power series expansion in the variable x of a framing dependent version of the Kauffman polynomial,

$$F^f\left(\diagdown\!\!\!\!\diagup\right) - F^f\left(\diagup\!\!\!\!\diagdown\right) = (e^{x/2} - e^{-x/2})\left(F^f\left(\,\rangle\,\langle\,\right) - F^f\left(\,\smile\!\!\!\!\frown\,\right)\right);$$
$$F^f\left(\uparrow\!\circlearrowright\right) = e^{\frac{N-1}{2}x}F^f\left(\,\uparrow\,\right). \quad \square$$

6.6. More on conjecture 2. Recalling that $\Phi \circ \bar{\sigma} = \tau \circ \mu \circ \bar{\sigma}$ and that $\bar{\sigma}$ and τ are isomorphisms (theorems 8 and 14), we see that conjecture 2 is true iff the map μ is one-to-one iff its adjoint μ^* is onto. Keeping in mind theorem 12, we see that conjecture 2 is essentially the assertion:

- Vassiliev invariants are precisely as powerful as the HOMFLY and Kauffman polynomials and all of their cablings.

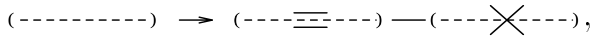
Whether or not this assertion is true, asking whether μ is 1-1 is clearly a simpler way of stating it, perhaps making it easier to resolve.

I have the following evidence to support conjecture 2:

- It had been verified up to degree 9.
- Its natural generalization had been verified for *homotopy link invariants*[15] (see [8]). A homotopy link invariant is a link invariant that does not change when an overcrossing in which only one component of the link is involved is replaced by an undercrossing.

[15]I wish to thank C. Day for teaching me about this class of invariants.

- Conjecture 2 is similar in form to the question of whether any weight system can be integrated to a knot invariant, which, after some hard work in section 4, was answered affirmatively. In both cases the question is whether a certain map is onto; in the case of conjecture 2 the map is the adjoint of

(41) 

while in the case of section 4 the map is the adjoint of

- To prove conjecture 2 one needs to show that it is possible to extend any functional ϕ defined on $\mathcal{B}$ to a functional (also called ϕ) defined on the space $\mathcal{B}^{pm}$ of partially marked diagrams — diagrams in which only a part of the arcs are marked — so that the extended ϕ maps (41) to an equality. It is not hard to show that over $\mathbf{Z}/2\mathbf{Z}$ it is possible to consistently extend every functional defined on unmarked diagrams to diagrams marked exactly once.

7. Odds and ends

7.1. Some questions.

7.1.1. *Why Lie algebras?* In some sense (theorem 1) studying the algebra $\mathcal{A}$ is *exactly* the same as studying Vassiliev invariants, and indeed, some of the structure of $\mathcal{A}$ can be understood in terms of knot theory (remark 3.2, exercises 3.6, 3.10, 3.14). The algebra $\mathcal{A}$ (or, in fact, $\mathcal{A}^*$) also has a (*weaker*) relation with Lie algebras (theorem 4 and conjecture 1), a relation which is not 1-1 (exercises 6.28 and 6.35) and not known to be onto. However, many of the results about $\mathcal{A}$ have a Lie theoretic interpretation but seem to have no knot theoretic interpretation.

Problem 7.1. Interpret theorems 6 and 8 and remark 5.7 in terms of knot theory.

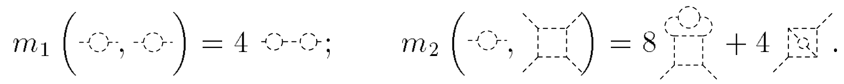
7.1.2. *Why surfaces.* A Short glance at the diagram in section 1.9 shows that to every marked surface naturally corresponds a knot invariant, and that the knot invariants thus obtained are rather strong.

Problem 7.2. Understand why do marked surfaces appear in knot theory. Find a direct topological construction for the invariant corresponding to a marked surface.

7.1.3. *The ground ring.*

Problem 7.3. How much of the theory of sections 4, 5 and 6 carries over to an arbitrary ground ring? A computer search (*e.g.* $\dim \mathcal{G}_m\mathcal{A}$ was re-computed over fields of small prime characteristics) suggests that perhaps everything carries over.

7.1.4. *Higher products.* On $\mathcal{B}$ (and therefore also on $\mathcal{A}$) there is a second grading: by the number of external vertices in a Chinese character. Furthermore, $\mathcal{B}$ has various *higher products*; for a positive integer j and a pair of Chinese characters C_1, C_2 define $m_j(C_1, C_2)$ to be the sum over all possible ways of sewing C_1 and C_2 along j external vertices:



Superficially, m_1 and the product of $\mathcal{A}$ are defined in similar ways. However, there appears to be no obvious relation between the two — their images are not even in the same degree!

Problem 7.4. Understand the second grading of $\mathcal{A}$ and the higher products m_j in terms of knot theory. Investigate the relations between the m_j and the other structures introduced in this paper.

7.2. Some bad news. The algebra $\mathcal{A}$ has a natural involution — simply map every chord diagram $D \in \mathcal{D}^c$ into the same diagram, only with the orientation of its circle reversed. In the language definition 3.11, this is just the operation ψ^{-1}, and therefore by exercise 3.14 it corresponds to the operation of orientation reversal for knots (reversal of the orientation of the knot itself, not of the ambient space!). Being an involution, ψ^{-1} ($(\psi^{-1})^\star$) decomposes $\mathcal{A}$ ($\mathcal{A}^\star$) into a sum of a $+1$ eigenspace and a -1 eigenspace. Call a diagram $D \in \mathcal{A}$ (a functional $W \in \mathcal{A}^\star$) *even* if it satisfies $\psi^{-1}D = D$ ($(\psi^{-1})^\star W = W$) in $\mathcal{A}$ ($\mathcal{A}^\star$), and call it *odd* if it satisfies $\psi^{-1}D = -D$ ($(\psi^{-1})^\star W = -W$).

Problem 7.5. Are there any odd diagrams? (This question was first raised by J. Birman)

A computer search has shown that all diagrams are even up to degree 9. This is rather disturbing because it means that all Vassiliev invariants up to degree 9 are even, *i.e.*, cannot tell a knot from its *inverse* (its orientation reversed version). Even worse than that is true — if conjecture 2 or even conjecture 1 is true, then *all* Vassiliev invariants are even, contradicting the hope implicitly expressed in problem 1.1:

Exercise 7.6. Show that via the isomorphism $\mathcal{A} \leftrightarrow \mathcal{B}$, an even (odd) diagram is carried to a Chinese character with an even (odd) number of external vertices.

Exercise 7.7. Without using theorems 11 or 12, show that if $C \in \mathcal{B}$ is a Chinese character with an odd number of external vertices, then $\Phi(C) = 0$.

Exercise 7.8. Show that if $\mathfrak{g}$ is a semi-simple Lie algebra and R is an arbitrary representation of $\mathfrak{g}$, then $W_{\mathfrak{g},R,m}$ is even for every m.

Hint 7.9. Use exercise 6.32 and the fact that every Lie algebra over the complex numbers has a conjugate-linear involution (*The Cartan involution*) that carries every representation to its complex conjugate.

7.3. Bibliographical remarks. In addition to the works mentioned in the body of this paper, the following papers also discuss Vassiliev invariants: J. C. Baez [4] (Vassiliev invariants for braids, relations with Chern-Simons theory and with quantum gravity) C. Day [14] (Vassiliev invariants of links), M. Gusarov [18], A. B. Sossinsky [38], T. Stanford [39] (Vassiliev invariants of links and graphs), and T. Stanford [40] (Vassiliev invariants for braids, examples for knots that cannot be separated apart by Vassiliev invariants of a fixed type).

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