8-25-1993

A calculus review.

I'm going to review some computational techniques from basic calculus.

If $z = f(x,y)$ is a function of two variables, the **partial derivative** of z (or f) with respect to x (denoted $\dfrac{\partial f}{\partial x}$) is obtained by differentiating with respect to x, holding y constant. Likewise, compute the partial derivative of f with respect to y (denoted $\dfrac{\partial f}{\partial y}$) by differentiating f with respect to y, holding x constant.

Example. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for $f(x,y) = \dfrac{x}{(x+y)^2}$.

$$\frac{\partial f}{\partial x} = \frac{(x+y)^2(1) - (x)(2)(x+y)}{(x+y)^4} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{2x}{(x+y)^3}. \ \square$$

Similarly, if $w = f(x,y,z)$, compute $\dfrac{\partial f}{\partial x}$ by differentiating f with respect to x, holding y and z constant (and similarly for $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$).

Example. Compute $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$ for $w = x^y \cdot z$.

$$\frac{\partial f}{\partial x} = yx^{y-1}z, \quad \frac{\partial f}{\partial y} = (\ln x)x^y \cdot z, \quad \frac{\partial f}{\partial z} = x^y.$$

Here is how I did $\dfrac{\partial f}{\partial y}$. Write $w = x^y \cdot z = e^{y \ln x} \cdot z$. Then

$$\frac{\partial f}{\partial y} = (\ln x)e^{y\ln x} \cdot z = (\ln x)x^y \cdot z. \ \square$$

For functions with several inputs and several outputs, the derivative is a *matrix* made up of the partial derivatives of the outputs with respect to the inputs. In particular, suppose

the inputs are $x_1, x_2, \ldots, x_m$ and

the outputs are $y_1, y_2, \ldots, y_n$.

That is, my function is

$$(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_m).$$

Then the derivative Df is

$$Df = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_m} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial y_n}{\partial x_1} & \dfrac{\partial y_n}{\partial x_2} & \cdots & \dfrac{\partial y_n}{\partial x_m} \end{bmatrix}.$$

The rows are labelled by the output variables and the columns by the input variables.

Example. Compute the derivative of $(u,v) = f(x,y,z) = (x^2 - y^2 + z^2, xy - xz + yz)$.

The notation means that
$$u = x^2 - y^2 + z^2 \quad \text{and} \quad xy - xz + yz.$$

So
$$Df = \begin{bmatrix} 2x & -2y & 2z \\ y - z & x + z & -x + y \end{bmatrix}. \quad \square$$

It's useful to set up **integration by parts** problems in tabular form, particularly when you have to do parts several times in succession.

Example. $\displaystyle \int e^{2x} \cos x \, dx$

Here's the parts table:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int \, dx \\ + & e^{2x} & & \cos x \\ & & \searrow & \\ - & 2e^{2x} & & \sin x \\ & & \searrow & \\ + & 4e^{2x} & \rightarrow & -\cos x \end{array}$$

There is no point in continuing, since I'll keep getting e^{2x}'s in the left column and sines and cosines in the right. Remember that to stop the computation at any point, you simply tack on the integral of the stuff in the last row. So I have
$$\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x \, dx.$$

Move the $4 \displaystyle\int e^{2x} \cos x \, dx$ back to the left:
$$5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x.$$

Divide by 5 and tack on the arbitrary constant:
$$\int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} \sin x + \frac{2}{5} e^{2x} \cos x + C. \quad \square$$

Integrals involving **even powers of sine and cosine** can be simplified using the **double angle formulas**:
$$(\cos\theta)^2 = \frac{1}{2}(1 + \cos 2\theta), \qquad (\sin\theta)^2 = \frac{1}{2}(1 - \cos 2\theta).$$

Example. $\displaystyle \int (\cos x)^4 \, dx$

$$\int (\cos x)^4 \, dx = \int \left(\frac{1}{2}(1 + \cos 2x)\right)^2 \, dx = \frac{1}{4} \int \left(1 + 2\cos 2x + (\cos 2x)^2\right) \, dx =$$
$$\frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \, dx = \frac{1}{4}\left(x + \sin 2x + \frac{x}{2} + \frac{1}{8} \sin 4x\right) + C =$$

$$\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \quad \square$$

If the integral contains an odd power of sine or cosine, it is better not to use the double angle formulas.

Example. $\int (\sin 5x)^3 \, dx$

$\int (\sin 5x)^3 \, dx = \int (\sin 5x)^2 (\sin 5x) \, dx =$	Pull off a sine.
$\int \left(1 - (\cos 5x)^2\right) (\sin 5x) \, dx =$	Use $(\sin\theta)^2 + (\cos\theta)^2 = 1$.
$-\frac{1}{5}\int (1 - u^2) \, du =$	Let $u = \cos 5x$.
$-\frac{1}{5}\left(u - \frac{1}{3}u^3\right) + C =$	Integrate.
$-\frac{1}{5}\cos 5x + \frac{1}{15}(\cos 5x)^3 + C$	Put the x's back. $\square$

Trig substitution converts a given integral to one involving trig functions, which may be easier to integrate.

Before I do an example, here are two useful integrals that you might want to save for future reference:

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int (\sec\theta)^3 \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

To do the first integral, multiply by $\dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$, then let $u = \sec\theta + \tan\theta$ to get $\displaystyle\int \frac{du}{u}$.

For the second integral, integrate by parts, apply the identity $1 + (\tan\theta)^2 = (\sec\theta)^2$, then find a copy of the original integral on the right side of the equation. Flip it back to the left, then divide by 2.

Example. $\int \sqrt{1 + x^2} \, dx$

The expression $1 + x^2$ looks like one side of the identity $1 + (\tan\theta)^2 = (\sec\theta)^2$, so I let $x = \tan\theta$. Then $dx = (\sec\theta)^2 \, d\theta$, and the integral becomes

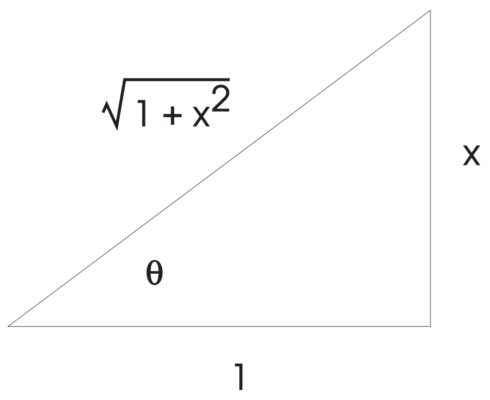
$$\int \sqrt{1 + x^2} \, dx = \int \sqrt{1 + (\tan\theta)^2} (\sec\theta)^2 \, d\theta = \int \sqrt{(\sec\theta)^2} (\sec\theta)^2 \, d\theta = \int (\sec\theta)^3 \, d\theta.$$

If you've made a sensible substitution, you should see a trig identity appear in the integral. (In this one, I had $1 + (\tan\theta)^2$, and replaced it with $(\sec\theta)^2$.) If you don't see an identity appear — perhaps after factoring a number out — then you've probably made the wrong substitution.

Using the integral I quoted prior to the example, I get

$$\int (\sec\theta)^3 \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

To put the x's back, draw a right triangle. The substitution was $\tan\theta = x = \dfrac{x}{1}$, and tangent is $\dfrac{\text{opposite}}{\text{adjacent}}$, so label the opposite side x and the adjacent side 1. The hypotenuse is $\sqrt{x^2 + 1}$ by Pythagoras.



Now $\tan\theta = x$ and $\sec\theta = \sqrt{x^2+1}$, so

$$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|\sqrt{x^2+1} + x| + C. \square$$