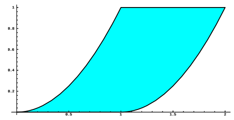
Change of variables.

I want to discuss the **change of variables** formula for multiple integrals, but first I'll give an example of a useful way to construct some coordinate transformations. This approach will work if you can decompose the target region into a union of (nonoverlapping) line segments.

Example. Let R be the region in the x-y plane bounded on the left by $y = x^2$, $0 \le x \le 1$, on the right by $y = (x-1)^2$, $1 \le x \le 2$, above by $y = 1$, and below by $y = 0$. Use horizontal segments to find a coordinate transformation which carries the unit square $0 \le u \le 1$, $0 \le v \le 1$ to R.

Here is the region:



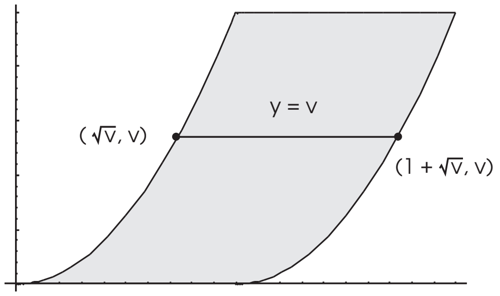
Notice that it can be broken up into horizontal segments. I will parametrize the segment at height $y = v$.

Recall that if P and Q are points, the line segment from P to Q can be parametrized by

$$(x, y) = (1-u) \cdot P + u \cdot Q, \quad 0 \le u \le 1.$$

You can derive this formula by finding the vector equation of the line through P and Q. However, it's easier to simply note that it has the right form for a line equation, and it passes through P (set $u = 0$) and Q (set $u = 1$) — so it must be the right thing.

The left hand curve is $y = x^2$. Set $y = v$: $x^2 = v$, so I get $x = \sqrt{v}$ — I want the positive square root, since the region is in the first quadrant. The left hand intersection point is $(\sqrt{v}, v)$.



The right hand curve is $y = (x-1)^2$. Set $y = v$: $(x-1)^2 = v$, so I get $x = 1 + \sqrt{v}$. The right hand intersection point is $(1 + \sqrt{v}, v)$.

Now I construct the segment joining the intersection points. By the discussion above, it is

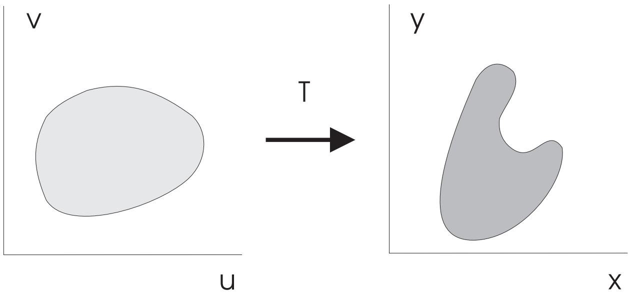
$$(x, y) = (1-u)(\sqrt{v}, v) + u(1 + \sqrt{v}, v) = (\sqrt{v} - u\sqrt{v}, v - uv) + (u + u\sqrt{v}, uv) = (u + \sqrt{v}, v).$$

What are the limits? For the segment parametrization I'm using, u always goes from 0 to 1. v is the height of the segment, and the region extends from 0 to 1.

Thus, the coordinate transformation is

$$x = u + \sqrt{v}, \quad y = v, \qquad 0 \le u \le 1, 0 \le v \le 1. \square$$

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation of the plane. Suppose T takes a region Q in the u-v plane to a region R in the x-y plane.



In component form, $(x, y) = T(u, v)$ looks like

$$x = (u \text{ and } v \text{ stuff}), \quad x = (\text{more } u \text{ and } v \text{ stuff}).$$

I need to assume make some technical assumptions to ensure that the transformation is reasonably "nice". Specifically:

1. T should have continuous first partial derivatives.
2. T should be **one-to-one** on Q. This means that different points of Q go to different points of R.
3. The Jacobian JT should be nonzero on Q.

You can usually get away with some exceptions to these conditions if the set of exceptional points is "small". For example, if $JT = 0$ on a curve (technically, on a set of **content** 0) the result is still true. To avoid technicalities, I'll only consider situations in which it is okay to proceed.

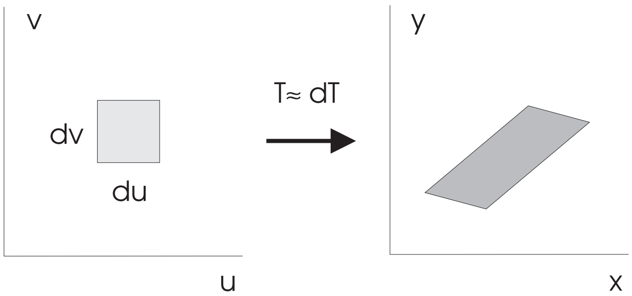
Under the assumptions above, the **change of variables** formula says that

$$\int\int_Q f\left[x(u, v), y(u, v)\right] \cdot JT\, du\, dv = \int\int_R f(x, y)\, dx\, dy.$$

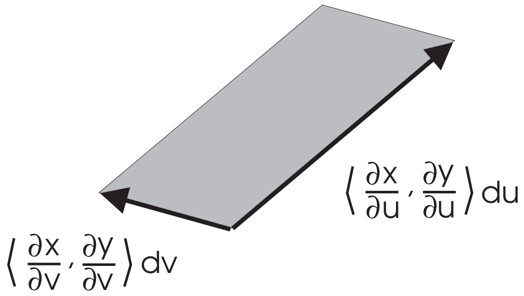
To apply the formula to compute $\displaystyle\int\int_R f(x, y)\, dx\, dy$, you need to:

1. Replace the limits for R with the limits for Q.
2. Use the transformation equations to substitute u and v for x and y.
3. Compute the Jacobian JT and insert it in the integral.

Steps 1 and 2 seem reasonable; what about step 3? The idea is that for a small region in the u-v plane, the picture looks like this:



For small increments of u and v, T is well approximated by the derivative dT. Apply dT to the sides $\langle du, 0\rangle$, $\langle 0, dv\rangle$ of the small rectangle in the u-v plane to get a small parallelogram in the x-y plane:



$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} \end{bmatrix} du, \qquad \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ dv \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \end{bmatrix} dv.$$

Thus, $\left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u} \right\rangle du$ and $\left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v} \right\rangle dv$ are the adjacent sides of the parallelogram. The parallelogram's area is

$$\left\| \begin{matrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{matrix} \right\| du\, dv = JT\, du\, dv.$$

That is, the Jacobian represents the factor by which small areas are scaled up by the transformation. This explains its presence in the change of variables formula.

Example. Let R be the region in the x-y plane bounded on the left by $y = x^2$, $0 \le x \le 1$, on the right by $y = (x-1)^2$, $1 \le x \le 2$, above by $y = 1$, and below by $y = 0$.

(a) $\displaystyle\iint_R (x^2 - y)\, dx\, dy$.

By the preceding example, the transformation

$$x = u + \sqrt{v}, \quad y = v, \qquad 0 \le u \le 1, 0 \le v \le 1$$

maps the unit square in the u-v plane onto R.

The Jacobian is

$$J = \left| \det \begin{bmatrix} 1 & \dfrac{1}{2\sqrt{v}} \\ 0 & 1 \end{bmatrix} \right| = 1.$$

The integrand is

$$x^2 - y = (u + \sqrt{v})^2 - v = u^2 + 2u\sqrt{v}.$$

Hence,

$$\iint_R (x^2 - y)\, dx\, dy = \int_0^1 \int_0^1 (u^2 + 2u\sqrt{v}) \cdot 1\, du\, dv = 1. \square$$

(b) Compute the area of R.

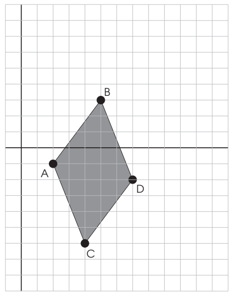
The area of R is given by the double integral $\displaystyle\iint_R dx\, dy$. Using the computations from part (a),

$$\iint_R dx\, dy = \int_0^1 \int_0^1 1\, du\, dv = 1.$$

(This is obvious if you remember Cavalieri's Principle from elementary geometry.) □

Example. Use a change of variables to compute $\displaystyle\iint_R (5x+2y-7)\,dx\,dy$, where R is the parallelogram with vertices $A(2,-1)$, $B(5,3)$, $C(4,-6)$, $D(7,-2)$.

A sketch of the parallelogram shows that B and C are adjacent to A:



$\overrightarrow{AB} = \langle 3, 4\rangle$ and $\overrightarrow{AC} = \langle 2, -5\rangle$, so the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

maps the square $0 \le u \le 1$, $0 \le v \le 1$ onto R. In component form, this is

$$x = 3u + 2v + 2, \quad y = 4u - 5v - 1.$$

The Jacobian is

$$\left|\det \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix}\right| = 23.$$

The integrand is

$$5x + 2y - 7 = 5(3u + 2v + 2) + 2(4u - 5v - 1) - 7 = 23u + 1.$$

Hence,

$$\iint_R (5x + 2y - 7)\,dx\,dy = \int_0^1 \int_0^1 (23u + 1) \cdot 23\,du\,dv = \frac{575}{2}. \ \square$$

Example. (a) Consider the coordinate transformation

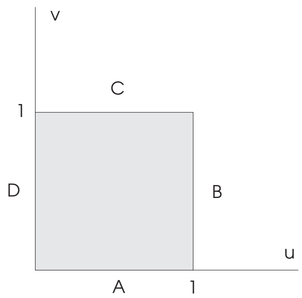
$$(x, y) = f(u, v) = (u^2 - v^2, u - v).$$

Find the image R in the x-y plane of the square $0 \le u \le 1$, $0 \le v \le 1$.

In component form the transformation is

$$x = u^2 - v^2, \quad y = u - v.$$

A reasonable way to find the image is to look at where the four sides of the square go.



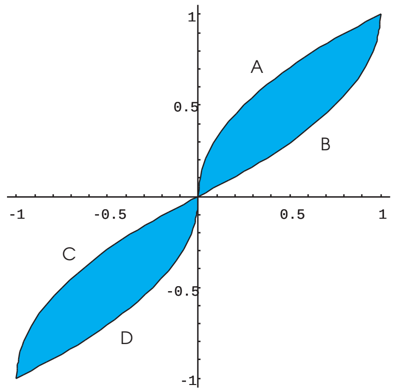
Look at side A: $0 \leq u \leq 1$, $v = 0$. The equations become $x = u^2$, $y = u$. Eliminating u, I get $x = y^2$. So this side of the square is mapped to the parabola $x = y^2$. Since $0 \leq u \leq 1$ and $y = u$, it is the part of the parabola which goes from $(0,0)$ to $(1,1)$.

Look at side B: $u = 1$, $0 \leq v \leq 1$. The equations become $x = 1 - v^2$, $y = 1 - v$. Then $v = 1 - y$, so $x = 1 - (1 - y)^2 = 2y - y^2$. This is a parabola; since $0 \leq v \leq 1$ and $y = 1 - v$, I have $0 \leq y \leq 1$. Hence, I want the part of the parabola which goes from $(0,0)$ to $(1,1)$.

Look at side C: $v = 1$, $0 \leq u \leq 1$. The equations become $x = u^2 - 1$, $y = u - 1$. Now $u = y + 1$, so $x = (y+1)^2 - 1 = y^2 + 2y$. This is a parabola again; since $0 \leq u \leq 1$ and $y = u - 1$, I want the part of the parabola which satisfies $-1 \leq y \leq 0$. It extends from $(0,0)$ to $(-1,-1)$.

Finally, look at side D: $u = 0$, $0 \leq v \leq 1$. The equations become $x = -v^2$, $y = -v$. Then $x = -y^2$, which is another parabola. Since $0 \leq v \leq 1$ and $y = -v$, I want the part of the parabola for which $-1 \leq y \leq 0$. It extends from $(0,0)$ to $(-1,-1)$.

The region R looks like two petals or leaves. I've drawn them below, with their edges labelled according to the sides of the square that mapped to them.



$\square$

(b) Is the transformation one-to-one on the square?

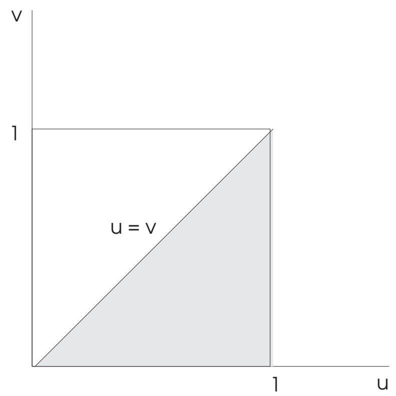
Do different imputs always produce different outputs? No — if $u = v$, then $x = 0$ and $y = 0$. That is, everything on the diagonal line $u = v$ is mapped to the origin in the x-y plane. It isn't one-to-one. $\square$

(c) Use the change of variables formula to compute $\displaystyle\int\int_R (x + y)\, dx\, dy$. Check by computing the integral directly. Does it work?

The Jacobian is

$$J = \left| \det \begin{bmatrix} 2u & -2v \\ 1 & -1 \end{bmatrix} \right| = |-2(u-v)|.$$

There are two cases. If $u - v \geq 0$, then $J = |-2(u-v)| = 2(u-v)$. If $u - v \leq 0$, then $J = |-2(u-v)| = -2(u-v)$. These two cases correspond to the halves of the square above and below the line $u = v$:



The bottom half (grey) is

$$0 \leq u \leq 1, \quad 0 \leq v \leq u \qquad (J = 2(u - v)).$$

The top half (white) is

$$0 \leq u \leq 1, \quad u \leq v \leq 1 \qquad (J = -2(u - v)).$$

The integrand is

$$x + y = u^2 - v^2 + u - v.$$

Therefore, the change of variables formula gives

$$\int\!\!\int_R (x+y)\,dx\,dy \stackrel{?}{=} \int_0^1 \int_0^u (u^2 - v^2 + u - v) \cdot (2(u - v))\,dv\,du + \int_0^1 \int_u^1 (u^2 - v^2 + u - v) \cdot (-2(u - v))\,dv\,du = 0.$$

Now I'll check by doing the integral as is, in x-y coordinates. The following inequalities describe the two petals:

$$\left\{ \begin{matrix} 0 \leq y \leq 1 \\ y^2 \leq x \leq 2y - y^2 \end{matrix} \right\} \qquad \left\{ \begin{matrix} -1 \leq y \leq 0 \\ 2y + y^2 \leq x \leq -y^2 \end{matrix} \right\}$$

So the integral is

$$\int_0^1 \int_{y^2}^{2y - y^2} (x + y)\,dx\,dy + \int_{-1}^0 \int_{2y + y^2}^{-y^2} (x + y)\,dx\,dy = 0.$$

The transformation is not one-to-one on the line $u = v$, and the Jacobian vanishes along that line as well, so I'm not *entitled* to assume it will work — but it does. $\square$