12-29-1993

Review - Test 1.

1. Let

$$f(x,y,z) = x(x+y-2z)^2 \quad \text{and} \quad \vec{F} = \langle x+yz, y-xz, z+xy \rangle.$$

Compute $\operatorname{grad} f$, $\operatorname{div} \vec{F}$, and $\operatorname{curl} \vec{F}$.

$$\operatorname{grad} f = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle (x+y-2z)^2 + 2x(x+y-2z), 2x(x+y-2z), -4x(x+y-2z) \rangle.$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+yz) + \frac{\partial}{\partial y}(y-xz) + \frac{\partial}{\partial z}(z+xy) = 1+1+1 = 3.$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+yz & y-xz & z+xy \end{vmatrix} =$$

$$\hat{\imath}\left(\frac{\partial}{\partial y}(z+xy) - \frac{\partial}{\partial z}(y-xz)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(z+xy) - \frac{\partial}{\partial z}(x+yz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(y-xz) - \frac{\partial}{\partial y}(x+yz)\right) =$$

$$\hat{\imath}(1-(-1)) - \hat{\jmath}(1-1) + \hat{k}(-1-1) = \langle 2, 0, -2 \rangle. \ \square$$

2. (a) Find the equation of the tangent plane to the surface

$$x = e^u \cos v, \quad y = e^{-u}, \quad z = e^u \sin v$$

at the point $\left(\dfrac{\sqrt{2}}{2}, 1, \dfrac{\sqrt{2}}{2}\right)$.

First, $\left(\dfrac{\sqrt{2}}{2}, 1, \dfrac{\sqrt{2}}{2}\right) = (x,y,z)$ gives $u = 0$ and $v = \dfrac{\pi}{4}$.

Next, I'll find the normal vector. I only need the normal at a single point, so I'll find $\overrightarrow{T_u}$ and $\overrightarrow{T_v}$ first.

$$\overrightarrow{T_u} = \langle e^u \cos v, -e^{-u}, e^u \sin v \rangle, \text{ so } \overrightarrow{T_u}\left(0, \frac{\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, -1, \frac{\sqrt{2}}{2} \right\rangle.$$

$$\overrightarrow{T_v} = \langle -e^u \sin v, 0, e^u \cos v \rangle, \text{ so } \overrightarrow{T_v}\left(0, \frac{\pi}{4}\right) = \left\langle -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right\rangle.$$

Therefore, the normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\sqrt{2}}{2} & -1 & \dfrac{\sqrt{2}}{2} \\ -\dfrac{\sqrt{2}}{2} & 0 & \dfrac{\sqrt{2}}{2} \end{vmatrix} = \left\langle -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2} \right\rangle.$$

The tangent plane is

$$\left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + (-1)(y-1) + \left(-\frac{\sqrt{2}}{2}\right)\left(z - \frac{\sqrt{2}}{2}\right) = 0. \ \square$$

(b) The surface

$$x = e^u \cos v, \quad y = e^{-u}, \quad z = e^u \sin v$$

can be generated by revolving a curve about one of the coordinate axes. What curve? What axis?

The surface can be generated by revolving the curve

$$x = e^u, \quad y = e^{-u},$$

about the y axis. The curve is the hyperbola $y = \dfrac{1}{x}$. □

3. Parametrize the surface which consists of the parallelogram with vertices $A(1,4,2)$, $B(3,5,3)$, $C(2,2,3)$, and $D(0,1,2)$.

First, I need to determine how the vertices are arranged. $\overrightarrow{AB} = \overrightarrow{DC} = \langle 2,1,1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle -1,-3,0\rangle$, but $\overrightarrow{AC} = \langle 1,-2,1\rangle$ and $\overrightarrow{BD} = \langle -3,-4,-1\rangle$. So the vertices B and D are adjacent to A.

The parametrization is

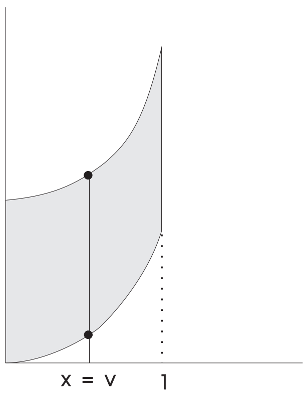
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix},$$

where $0 \le u \le 1$, $0 \le v \le 1$.

In component form, it is

$$x = -u + 2v + 1, \quad y = -3u + v + 4, \quad z = v + 2. \ \square$$

4. Let R be the region in the first quadrant bounded above by $y = e^x$, below by $y = x^2$, and on the right by $x = 1$. Find a coordinate transformation which maps the unit square $0 \le u \le 1$, $0 \le v \le 1$ onto R.



Consider a vertical line $x = v$, where $0 \le v \le 1$. The line intersects $y = x^2$ at the point (v, v^2). It intersects $y = e^x$ at the point (v, e^v). The segment joining (v, v^2) to (v, e^v) has parametric equations

$$(1-u)\cdot(v, v^2) + u\cdot(v, e^v), \quad 0 \le u \le 1.$$

If I multiply out, I find that the coordinate transformation is

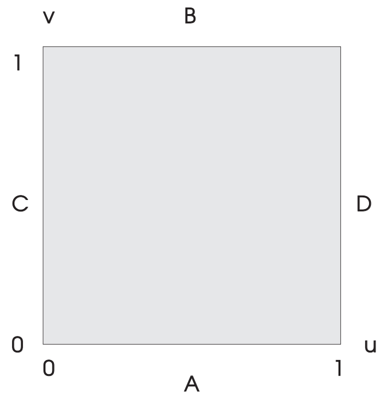
$$x = v, \quad y = v^2 - uv^2 + ue^v, \qquad 0 \le u \le 1, \quad 0 \le v \le 1. \ \square$$

5. What is the image of the unit square $0 \le u \le 1$, $0 \le v \le 1$ under the coordinate transformation

$$x = uv, \quad y = u - v?$$

The transformation takes the square to a region R in the x-y plane. The problem is to determine what R is.

I'll take the four sides of the square and find out where each goes.



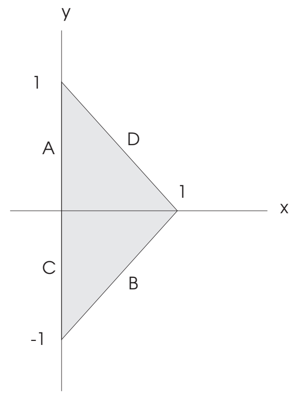
Side A: $v = 0$ gives $x = 0$ and $y = u$. For $0 \le u \le 1$, y goes from 0 to 1. This is a segment along the positive y axis.

Side B: $v = 1$ gives $x = u$ and $y = u - 1$, or $y = x - 1$. For $0 \le u \le 1$, I get the part of the line $y = x - 1$ for which $0 \le x \le 1$.

Side C: $u = 0$ gives $x = 0$ and $y = -v$. For $0 \le v \le 1$, y goes from 0 to -1. This is a segment along the negative y axis.

Side D: $u = 1$ gives $x = v$ and $y = 1 - v$, so $y = 1 - x$. For $0 \le v \le 1$, I get the part of the line $y = 1 - x$ for which $0 \le x \le 1$.

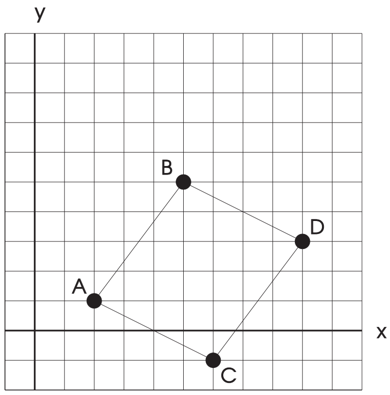
The region is shown in the picture below.



$\square$

6. Find a coordinate transformation which maps the unit square $0 \le u \le 1$, $0 \le v \le 1$, to the parallelogram R with vertices $A(2, 1)$, $B(5, 5)$, $C(6, -1)$, $D(9, 3)$. Then use it to evaluate

$$\int\!\!\int_R (2x + 2y)\, dx\, dy.$$



The vertices B and C are adjacent to A. $\overrightarrow{AB} = \langle 3, 4\rangle$ and $\overrightarrow{AC} = \langle 4, -2\rangle$, so the transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

In component form, this is

$$x = 3u + 4v + 2, \quad y = 4u - 2v + 1.$$

The integrand is

$$2x + 2y = (6u + 8v + 4) + (8u - 4v + 2) = 14u + 4v + 6.$$

The Jacobian is

$$J = \left|\det \begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}\right| = 22.$$

Hence,

$$\int\int_R (2x + 2y)\,dx\,dy = \int_0^1 \int_0^1 (14u + 4v + 6) \cdot 22\,du\,dv = 330. \square$$