11-22-1993

Class review - Test 3.

1. Solve the heat equation

$$u_t = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0$$

$$u(0,t) = 5, \quad u(2,t) = 11, \qquad t > 0$$

$$u(x,0) = 7x + 7$$

The solution has the form

$$u(x,t) = v(x) + w(x,t).$$

$v(x)$ is the steady-state solution and $w(x,t)$ is the transient solution.
To find $v(x)$, just draw a line connecting the boundary conditions $(0,5)$ and $(2,11)$. It is $v(x) = 3x+5$.
The transient solution $w(x,t)$ satisfies

$$w(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\pi^2 a^2 n^2 t}{L^2}\right) \sin\left(\frac{\pi n x}{L}\right).$$

The b_n's are the coefficients of the Fourier sine series for $u(x,0) - v(x) = (7x+7) - (3x+5) = 4x+2$. Compute the b_n's:

$$b_n = \frac{2}{2}\int_0^2 (4x+2)\sin\frac{\pi n x}{2}\,dx = \frac{16}{\pi^2 n^2}\left(\cos\pi n - 1\right).$$

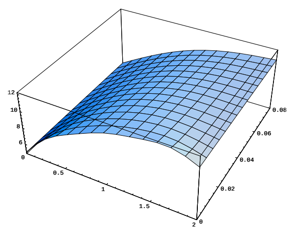
Hence,

$$w(x,t) = \sum_{n=1}^{\infty} \frac{16}{\pi^2 n^2}\left(\cos\pi n - 1\right)\exp(-\pi^2 n^2 t)\sin\frac{\pi n x}{2}.$$

The solution is

$$u(x,t) = 3x + 5 + w(x,t).$$

The picture below shows the graph of $u(x,t)$ during the first 0.08 time units.



Notice how the distribution approaches the linear steady-state distribution. $\square$

2. Solve the heat conduction problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0$$

$$u(x,0) = e^{-x}, \qquad 0 < x < 1$$

This the the rod with insulated ends ($u_x = 0$). The solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\pi^2 a^2 n^2 t}{L^2}\right) \cos\left(\frac{\pi n x}{L}\right).$$

The a_n's are the Fourier cosine coefficients of $u(x,0)$.
First,

$$a_0 = \frac{2}{1}\int_0^1 e^{-x}\,dx = 2(1 - e^{-1}).$$

Next,

$$a_n = \frac{2}{1}\int_0^1 e^{-x}\cos \pi n x\,dx = \frac{2\pi^2 n^2}{\pi^2 n^2 + 1}\left[e^{-x}\left(\frac{1}{\pi n}\sin \pi n x - \frac{1}{\pi^2 n^2}\cos \pi n x\right)\right]_0^1 = \frac{2}{\pi^2 n^2 + 1}\left(1 - e^{-1}\cos \pi n\right).$$

The solution is

$$u(x,t) = 1 - e^{-1} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2 + 1}\left(1 - e^{-1}\cos \pi n\right)\exp(-\pi^2 n^2 t)\cos \pi n x.$$

Here's the work for the integral.

$$\begin{array}{ccccc}
 & \dfrac{d}{dx} & & \int dx \\
+ & e^{-x} & & \cos \pi n x \\
 & & \searrow & \\
- & -e^{-x} & & \dfrac{1}{\pi n}\sin \pi n x \\
 & & \searrow & \\
+ & e^{-x} & & -\dfrac{1}{\pi^2 n^2}\cos \pi n x
\end{array}$$

So

$$\int e^{-x}\cos \pi n x\,dx = \frac{1}{\pi n}e^{-x}\sin \pi n x - \frac{1}{\pi^2 n^2}e^{-x}\cos \pi n x - \frac{1}{\pi^2 n^2}\int e^{-x}\cos \pi n x\,dx$$

$$\left(1 + \frac{1}{\pi^2 n^2}\right)\int e^{-x}\cos \pi n x\,dx = \frac{1}{\pi n}e^{-x}\sin \pi n x - \frac{1}{\pi^2 n^2}e^{-x}\cos \pi n x$$

$$\int e^{-x}\cos \pi n x\,dx = \frac{\pi^2 n^2}{\pi^2 n^2 + 1}e^{-x}\left(\frac{1}{\pi n}\sin \pi n x - \frac{1}{\pi^2 n^2}\cos \pi n x\right) + C \ \Box$$

3. Find the Fourier expansion of

$$f(x) = 1 - x, \quad -1 \le x < 1, \quad \text{and} \quad f(x) = f(x+2) \text{ for all } x.$$

First,

$$a_0 = \int_{-1}^1 (1 - x)\,dx = \left[x - \frac{1}{2}x^2\right]_{-1}^1 = 2.$$

Next,

$$c_n = \int_{-1}^1 (1 - x)e^{\pi n i x}\,dx = \left[\left(-\frac{i}{\pi n}(1 - x) - \frac{1}{\pi^2 n^2}\right)e^{\pi n i x}\right]_{-1}^1 =$$

$$\left(0-\frac{1}{\pi^2 n^2}\right)e^{\pi n i}-\left(-\frac{2i}{\pi n}-\frac{1}{\pi^2 n^2}\right)e^{-\pi n i}=\frac{2i}{\pi n}\cos \pi n.$$

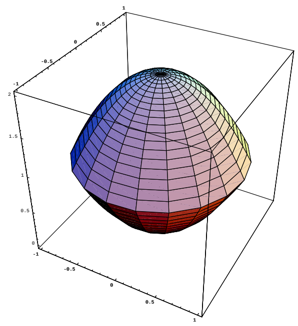
Thus,

$$a_n = \operatorname{re} c_n = 0, \quad b_n = \operatorname{im} c_n = \frac{2}{\pi n}\cos \pi n.$$

The Fourier series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{\pi n}\cos \pi n \sin(\pi n x). \ \Box$$

4. Let R be the region defined by the inequalities $x^2+y^2 \le z \le 2-x^2-y^2$. Find the flux of $\vec{F} = \left\langle \sqrt{x^2+y^2}, \sqrt{x^2+y^2}, (x^2+y^2)z \right\rangle$ out through the surface of R.



Use the Divergence Theorem. First,

$$\operatorname{div} \vec{F} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} + x^2 + y^2.$$

$x^2+y^2 = z = 2-x^2-y^2$ gives $x^2+y^2=1$, so the paraboloids intersect in $z=1$, $x^2+y^2=1$. Convert to cylindrical coordinates. The region is

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le 2-r^2.$$

Also,

$$\operatorname{div} \vec{F} = \cos\theta + \sin\theta + r^2.$$

So the flux is

$$\int_0^1 \int_{r^2}^{2-r^2} \int_0^{2\pi} (\cos\theta + \sin\theta + r^2) r \, d\theta \, dz \, dr = \int_0^1 \int_{r^2}^{2-r^2} 2\pi r^3 \, dz \, dr =$$

$$\int_0^1 2\pi r^3 (2-2r^2) \, dr = \frac{\pi}{3} \approx 1.04720.$$

(Incidentally, notice that doing θ first killed off some of the terms and made the rest of the computation easier.) $\Box$