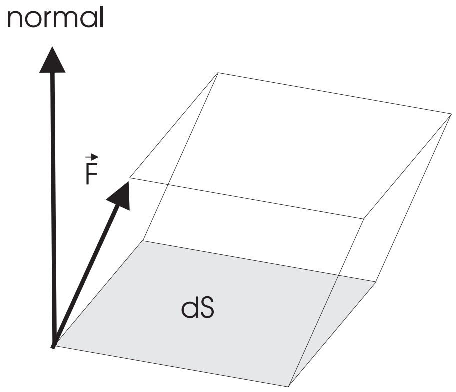
10-6-1993

Vector Surface Integrals.

If $\vec{F}$ is the velocity field of a fluid or a gas in space and S is a surface, the amount of fluid or gas which flows through a small piece dS of the surface is (roughly) the volume of the following parallelepiped:



The volume is (altitude) $\cdot\, dS$, and the altitude is

$$\operatorname{comp}_{\vec{N}} \vec{F} = \frac{\vec{F} \cdot \vec{N}}{|\vec{N}|}.$$

Hence, the volume is $\dfrac{\vec{F} \cdot \vec{N}}{|\vec{N}|}\, dS$.

However,

$$\overrightarrow{dS} = \vec{N}\, du\, dv = \frac{\vec{N}}{|\vec{N}|} |\vec{N}|\, du\, dv = \frac{\vec{N}}{|\vec{N}|}\, dS.$$

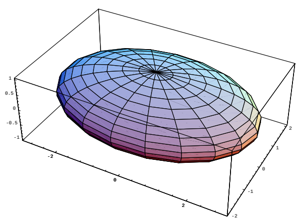
Therefore, the volume is $\vec{F} \cdot \overrightarrow{dS}$.

The total flow through S is therefore

$$\int\!\!\int_S \vec{F} \cdot \overrightarrow{dS}.$$

This is the **vector surface integral** of $\vec{F}$ over S, or the **flux** of $\vec{F}$ through S. Note that the sign of the integral depends on a choice of a normal to the surface. Usually, you will be told which normal vector to use.

Example. Find the flux of the radial vector field $\vec{F} = \langle x, y, z \rangle$ outward through the part of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ which lies in the first octant.



In the first octant, z is positive, so

$$z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}.$$

The normal vector is

$$\vec{N} = \left\langle \frac{\frac{cx}{a^2}}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}, \frac{\frac{cy}{b^2}}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}, 1 \right\rangle.$$

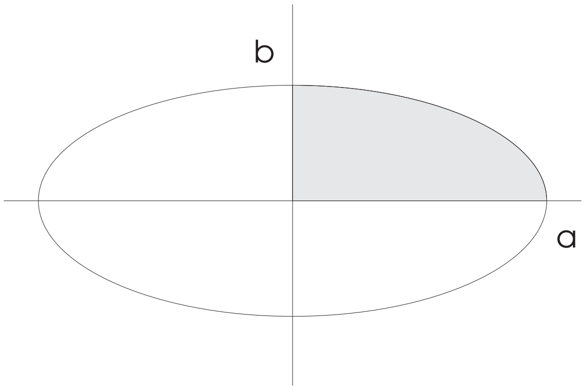
Therefore,

$$\vec{F} \cdot \vec{N} = \frac{\frac{cx^2}{a^2} + \frac{cy^2}{b^2}}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} + z = \frac{\frac{cx^2}{a^2} + \frac{cy^2}{b^2}}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} =$$

$$\frac{c}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{c}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}.$$

The projection of the surface into the x-y plane is given by the inequalities

$$0 \le x \le a, \quad 0 \le y \le b\sqrt{1 - \frac{x^2}{a^2}}.$$



Hence, the integral for the flux is

$$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{c}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\, dx\, dy.$$

To do the integral, use the change of variables

$$x = ra\cos\theta, \quad y = rb\sin\theta.$$

The inequalities for r and θ are

$$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

The Jacobian is

$$J = \left|\det\begin{bmatrix} a\cos\theta & b\sin\theta \\ -ra\sin\theta & rb\cos\theta \end{bmatrix}\right| = rab.$$

Therefore, the integral transforms to

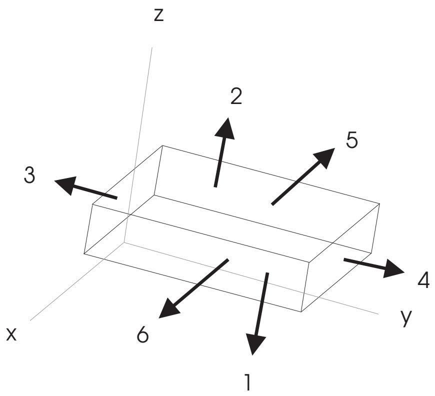
$$\int_0^{\pi/2} \int_0^1 \frac{rabc}{\sqrt{1-r^2}}\,dr\,d\theta = \frac{\pi abc}{2}. \ \square$$

Example. Let S be the surface of the box

$$1 \le x \le 2, \quad 1 \le y \le 3, \quad 2 \le z \le 3.$$

Find the flux of $\vec{F} = \langle x^2, y^2, z^2 \rangle$ out through S.

I simply compute the flux through each side. I've labelled the sides in the picture below.



Side 1 is $z = 2$ for $1 \le x \le 2$ and $1 \le y \le 3$. The normal would be

$$\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle 0, 0, 1 \rangle.$$

But from the picture, you can see that this points *into* the box. So instead, I use $\vec{N} = \langle 0, 0, -1 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 0, -1 \rangle = -z^2 = -4.$$

($-z^2 = -4$ because $z = 2$.) The flux is

$$\int_1^2 \int_1^3 -4\,dx\,dy = -8.$$

Side 2 is $z = 3$ for $1 \le x \le 2$ and $1 \le y \le 3$. The normal is $\vec{N} = \langle 0, 0, 1 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 0, 1 \rangle = z^2 = 9.$$

The flux is

$$\int_1^2 \int_1^3 9\,dx\,dy = 18.$$

Side 3 is $y = 1$ for $1 \le x \le 2$ and $2 \le z \le 3$. The normal is $\vec{N} = \langle 0, -1, 0 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle 0, -1, 0 \rangle = -y^2 = -1.$$

The flux is

$$\int_1^2 \int_2^3 (-1)\,dx\,dy = -1.$$

Side 4 is $y = 3$ for $1 \le x \le 2$ and $2 \le z \le 3$. The normal is $\vec{N} = \langle 0, 1, 0 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle 0, 1, 0 \rangle = y^2 = 9.$$

The flux is

$$\int_1^2 \int_2^3 9 \, dx \, dy = 9.$$

Side 5 is $x = 1$ for $1 \le y \le 3$ and $2 \le z \le 3$. The normal is $\vec{N} = \langle -1, 0, 0 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle -1, 0, 0 \rangle = -x^2 = -1.$$

The flux is

$$\int_1^3 \int_2^3 (-1) \, dx \, dy = -2.$$

Side 6 is $x = 2$ for $1 \le y \le 3$ and $2 \le z \le 3$. The normal is $\vec{N} = \langle 1, 0, 0 \rangle$.

$$\vec{F} \cdot \vec{N} = \langle x^2, y^2, z^2 \rangle \cdot \langle 1, 0, 0 \rangle = x^2 = 4.$$

The flux is

$$\int_1^3 \int_2^3 4 \, dx \, dy = 8.$$

The total flux is

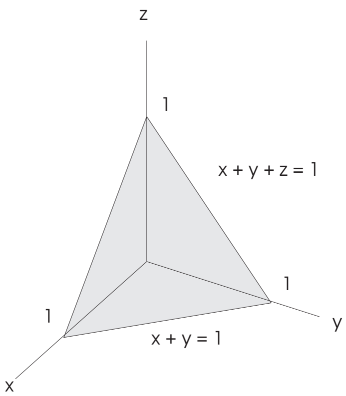
$$-8 + 18 + (-1) + 9 + (-2) + 8 = 24.$$

Notice that the flux of a vector field out of a closed region need not be 0 (just as the line integral of a vector field around a closed curve need not be 0). □

Example. Compute

$$\int\int_S xz \, dx \, dy + xy \, dy \, dz + yz \, dx \, dz,$$

where S is the part of $x + y + z = 1$ lying in the first quadrant.



I will do each of the terms separately. First,

$$\int\int_S xz \, dx \, dy = \int\int_{R_{xy}} xz \, dx \, dy,$$

where R_{xy} is the projection of the surface into the x-y plane. (I project into the x-y plane because the differentials are $dx \, dy$.) Now $z = 1 - x - y$, and the projection is

$$0 \le x \le 1, \quad 0 \le y \le 1 - x.$$

So

$$\iint_{R_{xy}} xz\,dx\,dy = \int_0^1 \int_0^{1-x} x(1-x-y)\,dy\,dx = \frac{1}{24}.$$

Similarly, if R_{yz} and R_{xz} are the projections into the y-z and x-z planes, respectively, then

$$\iint_S xy\,dy\,dz = \int_0^1 \int_0^{1-y} (1-y-z)y\,dz\,dy = \frac{1}{24}$$

$$\iint_S yz\,dx\,dz = \int_0^1 \int_0^{1-x} (1-x-z)z\,dz\,dx = \frac{1}{24}$$

The total is $\dfrac{1}{24}+\dfrac{1}{24}+\dfrac{1}{24}=\dfrac{1}{8}$. □

Example. Compute the flux of $\vec{F}=\langle x,y,z\rangle$ *upward* through the triangle with vertices $A(1,2,0)$, $B(2,1,-1)$, and $C(1,1,1)$.

$\overrightarrow{AB}=\langle 1,-1,-1\rangle$ and $\overrightarrow{AC}=\langle 0,-1,1\rangle$, so the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

maps the triangle $0\le u\le 1$, $0\le v\le 1-u$ onto triangle ABC.

In component form, this is

$$x=u+1,\quad y=-u-v+2,\quad z=-u+v.$$

The normal vector is

$$\overrightarrow{T_u}\times\overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = \langle -2,-1,-1\rangle.$$

But the problem asks for the flux *upward*, so I need the normal with positive z-component. It is $\langle 2,1,1\rangle$. Then

$$\vec{F}\cdot -(\overrightarrow{T_u}\times\overrightarrow{T_v}) = \langle u+1,-u-v+2,-u+v\rangle\cdot\langle 2,1,1\rangle = 4.$$

The flux is

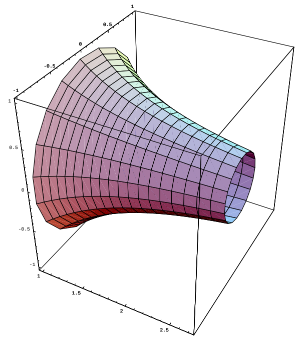
$$\int_0^1 \int_0^{1-u} 4\,dv\,du = 2.$$

(The flux *downward* through the triangle would be -2.) □

Example. The surface

$$x=e^u,\quad y=e^{-u}\cos v,\quad z=e^{-u}\sin v,$$

$0\le u\le 1$, $0\le v\le 2\pi$, is generated by revolving the curve $x=e^u$, $y=e^{-u}$, $0\le u\le 1$, about the x-axis. I'll find the flux of $\vec{F}=\langle -x^2,z,y\rangle$ *out through* the surface.



The normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^u & -e^{-u}\cos v & -e^{-u}\sin v \\ 0 & -e^{-u}\sin v & e^{-u}\cos v \end{vmatrix} = \langle -e^{-2u}, -\cos v, -\sin v \rangle.$$

So

$$\vec{F} \cdot (\overrightarrow{T_u} \times \overrightarrow{T_v}) = \langle -e^{2u}, e^{-u}\sin v, e^{-u}\cos v \rangle \cdot \langle -e^{-2u}, -\cos v, -\sin v \rangle = 1 - 2e^{-u}\sin v \cos v.$$

The flux is

$$\int_0^1 \int_0^{2\pi} (1 - 2e^{-u}\sin v \cos v)\, dv\, du = 2\pi. \square$$