Math 227
12-9-1993

Review Session for the Final - 1.

The Final is Tuesday, December 14, 3:30 - 6:30 p.m., in Bingham 304 and Bingham 305. (Sit in alternate seats, and use whichever room has space.)

The remaining Review Sessions are Sunday, December 12 and Monday, December 13, both at 7:30 p.m. in Bingham 103.

The remaining ungraded homework will be done by Friday, December 10, at 2 p.m. All the review sheets are done; there were two of them, and there are copies in the box outside my office.

1. Solve the heat conduction problem

$$u_t = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0$$

$$u(0,t) = 2, \quad u(2,t) = 8, \qquad t > 0$$

$$u(x,0) = 5x, \qquad 0 < x < 2$$

$u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady state solution and $w(x,t)$ is the transient solution. $v(x)$ is the line through $(0,2)$ and $(2,8)$: $v(x) = 3x + 2$. Then

$$w(x,0) = u(x,0) - v(x) = 5x - (3x+2) = 2x - 2.$$

Find the Fourier sine coefficients for $w(x,0)$:

$$b_n = \frac{2}{2}\int_0^2 (2x-2)\sin\frac{\pi n x}{2}\,dx = \left[-\frac{2}{\pi n}(2x-2)\cos\frac{\pi n x}{2} + \frac{8}{\pi^2 n^2}\sin\frac{\pi n x}{2}\right]_0^2 = -\frac{4}{\pi n}(1+\cos(\pi n)).$$

Therefore,

$$w(x,t) = \sum_{n=1}^{\infty} -\frac{4}{\pi n}(1+\cos(\pi n))\exp(-\pi^2 n^2 t)\sin\frac{\pi n x}{2}.$$

The solution is $u(x,t) = 3x + 2 + w(x,t)$. $\square$

2. Compute the circulation of

$$\vec{F} = \langle xz + \tan x - y^3, -yz + \tan y + x^3, (\sec z)^2\rangle$$

around the curve

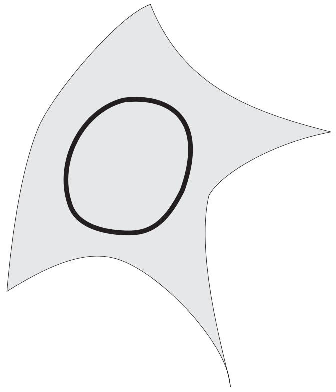
$$\vec{\sigma}(t) = \langle \cos t, \sin t, \sin 2t\rangle, \qquad 0 \le t \le 2\pi.$$

Assume that the curve is traversed counterclockwise as viewed from above.

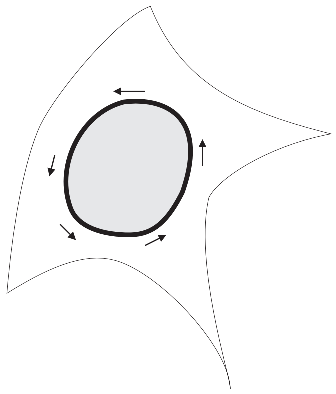
It's natural to try to use Stokes' theorem. First,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz + \tan x - y^3 & -yz + \tan y + x^3 & (\sec z)^2 \end{vmatrix} = \langle y, x, 3x^2 + 3y^2\rangle.$$

Now I need to find a surface bounded by $\vec{\sigma}(t)$. The idea is to find a surface which contains the curve:



Then the part of the surface "inside" the curve is a surface having the curve as its boundary:



To find a surface which contains the curve, I need to find a surface equation which is satisfied by $x = \cos t$, $y = \sin t$, and $z = \sin 2t$. Here's a familiar relationship between $\cos t$, $\sin t$, and $\sin 2t$:

$$\sin 2t = 2 \sin t \cos t.$$

Since the curve equations satisfy $z = 2xy$, the curve lies in the surface $z = 2xy$. Since $x = \cos t$ and $y = \sin t$, if I project the part of the surface bounded by $\vec{\sigma}$ into the x-y plane, I get the disk $x^2 + y^2 \leq 1$. Convert to polar: I will integrate over

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

For this curve, the normal is

$$\vec{N} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle -2y, -2x, 1 \rangle.$$

Then

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -2y^2 - 2x^2 + 3x^2 + 3y^2 = x^2 + y^2 = r^2.$$

By Stokes' theorem, the circulation around $\vec{\sigma}$ is

$$\int_0^{2\pi} \int_0^1 r^2 \cdot r \, dr \, d\theta = \frac{\pi}{2} \approx 1.57080. \square$$

3. Let

$$\vec{F} = \langle 2xy^2 - 3z, 2x^2y, -3x + 3z^2 \rangle.$$

Compute $\int \vec{F} \cdot \overrightarrow{ds}$, where

$$\vec{\sigma}(t) = \left\langle te^{t-1}, t^2, 2t^3 \right\rangle, \qquad 0 \le t \le 1.$$

$$\operatorname{curl} \vec{F} = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy^2 - 3z & 2x^2y & -3x + 3z^2 \end{array} \right| = \vec{0}.$$

Therefore, the field is conservative.
I will find a potential function. If $\vec{F} = \nabla f$, then

$$\frac{\partial f}{\partial x} = 2xy^2 - 3z, \quad \frac{\partial f}{\partial y} = 2x^2y, \quad \frac{\partial f}{\partial z} = -3x + 3z^2.$$

Integrate $\dfrac{\partial f}{\partial x}$ with respect to x:

$$f = x^2y^2 - 3xz + C(y, z).$$

Differentiate with respect to y:

$$2x^2y = \frac{\partial f}{\partial y} = 2x^2y + \frac{\partial C}{\partial y}.$$

Now $\dfrac{\partial C}{\partial y} = 0$, so $C = D(z)$. Thus,

$$f = x^2y^2 - 3xz + D(z).$$

Differentiate with respect to z:

$$-3x + 3z^2 = \frac{\partial f}{\partial z} = -3x + \frac{dD}{dz}.$$

Then $\dfrac{dD}{dz} = 3z^2$, so $D = z^3$. Hence,

$$f = x^2y^2 - 3xz + z^3.$$

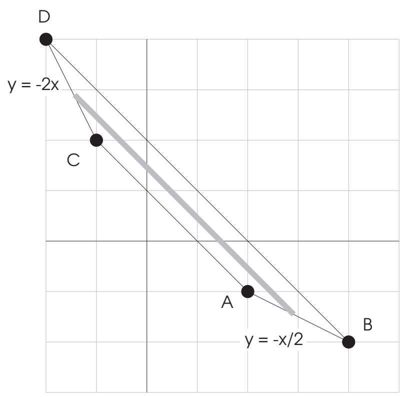
Now $\vec{\sigma}(0) = (0, 0, 0)$ while $\vec{\sigma}(1) = (1, 1, 2)$, so

$$\int \vec{F} \cdot \overrightarrow{ds} = f\left[\vec{\sigma}(1)\right] - f\left[(0, 0, 0)\right] = 3. \ \square$$

4. Let R be the quadrilateral with vertices $A(2, -1)$, $B(4, -2)$, $C(-1, 2)$, and $D(-2, 4)$.

(a) Find a coordinate transformation that maps a rectangle in the u-v plane onto R.

I'll break the region up into diagonal segments. A typical segment is shown in grey in the picture below.



The bottom end of the segment is on the line $y = -\frac{1}{2}x$, where $2 \le x \le 4$. Let $x = u$. Then a typical point is $\left(u, -\frac{1}{2}u\right)$.

Next, I'll find the equation of the grey segment. It clearly has slope -1, so it is $y = -x + a$ for some a. It passes through $\left(u, -\frac{1}{2}u\right)$, so $-\frac{1}{2}u = -u + a$, or $a = \frac{1}{2}u$. Thus, the x-y equation of the grey segment is $y = -x + \frac{1}{2}u$.

What is the top end of the segment? It is the point where the segment hits the line $y = -2x$. Solve $y = -2x$ simultaneously with $y = -x + \frac{1}{2}u$; then $x = -\frac{1}{2}u$, so $y = -2x = u$. The top end is $\left(-\frac{1}{2}u, u\right)$. (You could have seen this by symmetry.)

Now I just construct the segment joining $\left(u, -\frac{1}{2}u\right)$ to $\left(-\frac{1}{2}u, u\right)$:

$$(1 - v) \cdot \left(-\frac{1}{2}u, u\right) + v \cdot \left(u, -\frac{1}{2}u\right) = \left(-\frac{1}{2}u + \frac{3}{2}uv, u - \frac{3}{2}uv\right).$$

The coordinate transformation is therefore

$$x = -\frac{1}{2}u + \frac{3}{2}uv, \quad y = u - \frac{3}{2}uv, \qquad 2 \le u \le 4, \quad 0 \le v \le 1. \square$$

(b) Use the coordinate transformation to compute

$$\iint_R (x + y)\, dx\, dy.$$

$x + y = \frac{1}{2}u$. The Jacobian is

$$\left|\det \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}v & \frac{3}{2}u \\ 1 - \frac{3}{2}v & -\frac{3}{2}u \end{bmatrix}\right| = \frac{3}{2}u.$$

Hence,

$$\iint_R (x + y)\, dx\, dy = \int_0^1 \int_2^4 \frac{1}{2}u \cdot \frac{3}{2}u\, du\, dv = 7. \square$$

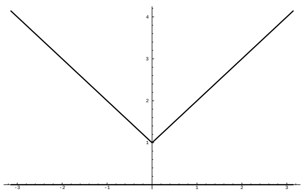
5. Find the Fourier expansion of

$$f(x) = |x| + 1, \qquad -\pi \le x < \pi, \quad \text{and} \quad f(x) = f(x + 2\pi) \text{ for all } x.$$

Write the function as

$$f(x) = \begin{cases} x + 1 & \text{if } 0 < x \le \pi \\ -x + 1 & \text{if } -\pi \le x < 0 \end{cases}$$

The function is even as you can see by drawing the graph.



Therefore, there will be no sine terms in the expansion, and I'll just compute the cosine coefficients. First,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} (-x+1)\, dx + \int_{0}^{\pi} (x+1)\, dx \right) = 2 + \pi.$$

Next

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \left(\int_{-\pi}^{0} (-x+1) \cos nx\, dx + \int_{0}^{\pi} (x+1) \cos nx\, dx \right) = -\frac{2}{\pi n^2} \left(1 - \cos(\pi n)\right).$$

Therefore,

$$f(x) = \frac{1}{2}(2+\pi) - \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left(1 - \cos(\pi n)\right) \cos nx. \quad \Box$$