Review Session for the Final - 2.

1. A field $\vec{F}$ such that $\operatorname{div}\vec{F} = 0$ is called **solenoidal**. If $\vec{F}$ is a vector field whose components have continuous partial derivatives and $\operatorname{div}\vec{F} = 0$, there is a vector field $\vec{G}$ such that $\vec{F} = \operatorname{curl}\vec{G}$.

In fact, if $\vec{F} = \langle F_1, F_2, F_3\rangle$ is solenoidal, then $\vec{G} = \langle G_1, G_2, G_3\rangle$ is given by

$$G_1 = \int_0^z F_2(x, y, t)\,dt - \int_0^y F_3(x, t, 0)\,dt$$

$$G_2 = -\int_0^z F_1(x, y, t)\,dt$$

$$G_3 = 0$$

It may seem wrong that the third component is always 0, but there's no problem. If you have a $\vec{G}$ that works and you take any function f, then

$$\operatorname{curl}(\nabla f + \vec{G}) = \operatorname{curl}\operatorname{grad} f + \operatorname{curl}\vec{G} = \vec{0} + \vec{F} = \vec{F}.$$

That is $\nabla f + \vec{G}$ also works. So you could get a solution with a nonzero third component by simply tacking on a gradient field with a nonzero third component.

For example, let $\vec{F} = x^2 + z, -2xy - y, z\rangle$. Then

$$\operatorname{div}\vec{F} = 2x - 2x - 1 + 1 = 0.$$

$\vec{F}$ is solenoidal. Use the formulas above to find $\vec{G}$ such that $\operatorname{curl}\vec{G} = \vec{F}$.

$F_2(x, y, z) = 2xy - y$, so $F_2(x, y, t) = -2xy - y$. $F_3(x, y, z) = z$, so $F_3(x, t, 0) = 0$. Therefore,

$$G_1 = \int_0^z F_2(x, y, t)\,dt - \int_0^y F_3(x, t, 0)\,dt = \int_0^z (-2xy - y)\,dt - 0 = -2xyz - yz.$$

Next, $F_1(x, y, z) = x^2 + z$, so $F_1(x, y, t) = x^2 + t$. Therefore,

$$G_2 = -\int_0^z F_1(x, y, t)\,dt = -\int_0^z (x^2 + t)\,dt = -x^2 z - \frac{1}{2}z^2.$$

Finally, $G_3 = 0$. Therefore,

$$\vec{G} = \left\langle -2xyz - yz, -x^2 z - \frac{1}{2}z^2, 0\right\rangle. \quad \Box$$

2. Consider the coordinate transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

(a) What set of points in the u-v plane maps to the square $0 \le x \le 2$, $0 \le y \le 2$?

(b) What is the image of the square $0 \le u \le 1$, $0 \le v \le 1$?

(a) Invert the transformation to find u and v in terms of x and y:

$$\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ -1 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 5x - 3y - 21 \\ -x + y + 5 \end{bmatrix}.$$

Now affine transformations (such as the one in this problem) map lines to lines. So I just need to find the four points which map to the vertices of the square. I plug the x and y coordinates into the formula above and find the correspondig u and v coordinates.

(x, y)	(u, v)
$(0, 0)$	$\left(-\frac{21}{2}, \frac{5}{2}\right)$
$(2, 0)$	$\left(-\frac{11}{2}, \frac{3}{2}\right)$
$(0, 2)$	$\left(-\frac{27}{2}, \frac{7}{2}\right)$
$(2, 2)$	$\left(-\frac{17}{2}, \frac{5}{2}\right)$

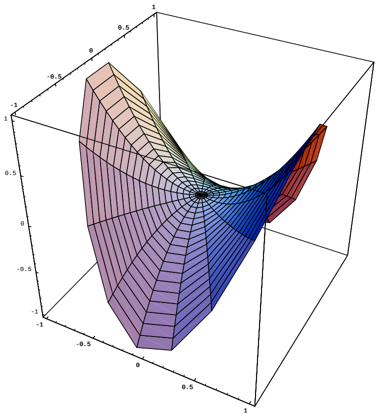
The parallelogram with vertices $\left(-\frac{21}{2}, \frac{5}{2}\right)$, $\left(-\frac{11}{2}, \frac{3}{2}\right)$, $\left(-\frac{27}{2}, \frac{7}{2}\right)$, $\left(-\frac{17}{2}, \frac{5}{2}\right)$ maps to the square $0 \le x \le 2$, $0 \le y \le 2$. $\square$

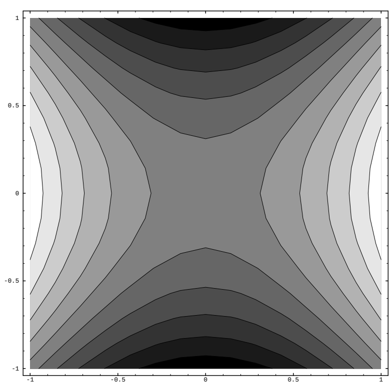
(b) Simply plug the vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ (for u and v) into the original transformation. For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

The image of the square $0 \le u \le 1$, $0 \le v \le 1$ is the parallelogram with vertices $(3, -2)$, $(4, -1)$, $(6, 3)$, $(7, 4)$. $\square$

3. Find the mass of the part of the surface $z = \frac{1}{2}(x^2 - y^2)$ which lies inside the cylinder $x^2 + y^2 = 1$, if the density is $\delta = |z|$.





By symmetry, I can find the mass of the part of the surface in the range $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$ and multiply by 4. On this part of the surface $z \ge 0$, so the density is simply

$$\delta = |z| = z = x^2 - y^2.$$

Convert to polar: $\delta = r^2\left[(\cos\theta)^2 - (\sin\theta)^2\right]$.
The normal is

$$\vec{N} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle = \langle -x, y, 1\rangle \quad \text{so} \quad |\vec{N}| = \sqrt{x^2+y^2+1} = \sqrt{r^2+1}.$$

I want the part of the surface inside the cylinder $x^2 + y^2 = 1$, i.e. $r = 1$, so $0 \le r \le 1$. As noted above, $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$. Therefore, the mass is

$$4\int_{-\pi/4}^{\pi/4}\int_0^1 r^2\left[(\cos\theta)^2 - (\sin\theta)^2\right]\sqrt{r^2+1}\cdot r\,dr\,d\theta = \frac{8}{15}(\sqrt{2}+1) \approx 1.28758. \ \square$$

4. (a) If P is a surface, a scalar surface integral over P looks like

$$\iint_P f(x,y,z)\,dS.$$

A vector surface integral over P looks like

$$\iint_P \vec{F}(x,y,z)\cdot\overrightarrow{dS}.$$

Suppose $\vec{n}$ is a *unit normal* to P. Show that

$$\overrightarrow{dS} = \vec{n}dS, \quad \text{and hence} \iint_P \vec{F}(x,y,z)\cdot\overrightarrow{dS} = \iint_P \vec{F}(x,y,z)\cdot\vec{n}\,dS.$$

If P is parametrized by u and v, then

$$\overrightarrow{dS} = (\overrightarrow{T_u}\times\overrightarrow{T_v})\,du\,dv \quad \text{whereas} \quad dS = |\overrightarrow{T_u}\times\overrightarrow{T_v}|\,du\,dv.$$

Now $\overrightarrow{T_u}\times\overrightarrow{T_v}$ is a normal to the surface, so

$$\vec{n} = \frac{\overrightarrow{T_u}\times\overrightarrow{T_v}}{|\overrightarrow{T_u}\times\overrightarrow{T_v}|}.$$

Hence,

$$\vec{n}dS = \frac{\overrightarrow{T_u}\times\overrightarrow{T_v}}{|\overrightarrow{T_u}\times\overrightarrow{T_v}|}\cdot|\overrightarrow{T_u}\times\overrightarrow{T_v}|\,du\,dv = (\overrightarrow{T_u}\times\overrightarrow{T_v})\,du\,dv = \overrightarrow{dS}.$$

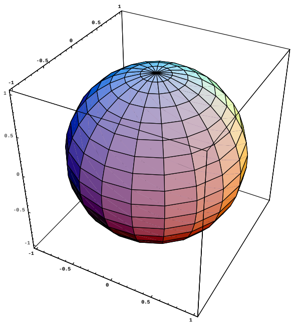
$$\iint_P \vec{F}(x,y,z)\cdot\overrightarrow{dS} = \iint_P \vec{F}(x,y,z)\cdot\vec{n}\,dS$$

follows by substitution. $\square$

(b) Use the Divergence Theorem to evaluate

$$\iint_P (x^2 - y^2 + z^2)\, dS$$

where P is the unit sphere $x^2 + y^2 + z^2 = 1$.



Note that P is a closed surface; it bounds the solid ball $x^2 + y^2 + z^2 \le 1$. By (a), the given integral will be in Divergence Theorem form if I can write the integrand $x^2 - y^2 + z^2$ as $\vec{F} \cdot \vec{n}$, where $\vec{F}$ is a vector field and $\vec{n}$ is the unit vector field for the sphere.

$\vec{n} = \langle x, y, z\rangle$. Why? Well, at a point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$, the vector $\langle x, y, z\rangle$ points radially out from the origin, so it has to be perpendicular to the sphere. And since $x^2 + y^2 + z^2 = 1$ for points on the sphere, $|\vec{n}| = \sqrt{x^2 + y^2 + z^2} = 1$.

(Dirty trick, huh?)

Now I need $\vec{F}$ such that

$$\vec{F} \cdot \langle x, y, z\rangle = x^2 - y^2 + z^2.$$

Obvious choice: $\vec{F} = \langle x, -y, z\rangle$.

Now apply the Divergence Theorem.

$$\operatorname{div} \vec{F} = 1 - 1 + 1 = 1.$$

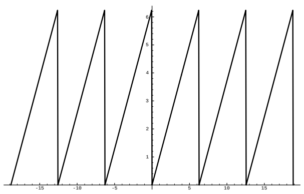
So

$$\iint_P (x^2 - y^2 + z^2)\, dS = \iiint_{x^2+y^2+z^2\le 1} (1)\, dv = (\text{ the volume of } x^2 + y^2 + z^2 \le 1) = \frac{4}{3}\pi \approx 4.18879. \square$$

5. Find the Fourier expansion of

$$f(x) = x, \qquad 0 \le x < 2\pi, \quad \text{and} \quad f(x) = f(x + 2\pi) \text{ for all } x.$$

Note that this function is periodic with period 2π. Here is its graph:



If you want to use the formula

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x)\exp\frac{\pi n i x}{L}\,dx$$

with $L = \pi$, you need to find an expression for f on $-\pi \le x < \pi$. It is

$$f(x) = \begin{cases} x + 2\pi & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x < \pi \end{cases}$$

But because f repeats every 2π units, the integral from $-\pi$ to 0 is the same as the integral from π to 2π. So instead, I can use

$$c_n = \frac{1}{L}\int_{0}^{2L} f(x)\exp\frac{\pi n i x}{L}\,dx$$

First,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = 2\pi.$$

Next,

$$c_n = \frac{1}{\pi}\int_0^{2\pi} x e^{nix}\,dx = \frac{1}{\pi}\left[\left(-\frac{i}{n}x + \frac{1}{n^2}\right)e^{nix}\right]_0^{2\pi} = -\frac{2i}{n}.$$

Hence, $a_n = 0$ and $b_n = -\dfrac{2}{n}$. The Fourier expansion is

$$f(x) = \pi + \sum_{n=1}^{\infty} -\frac{2}{n}\sin(nx). \square$$