Math 227
12-26-1993

Review Session for the Final - 3.

1. (a) Parametrize the triangle with vertices $A(1,1,1)$, $B(3,1,2)$, $C(2,3,2)$.

$\overrightarrow{AB} = \langle 2,0,1\rangle$ and $\overrightarrow{AC} = \langle 1,2,1\rangle$. The triangle is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The limits on the parameters are

$$0 \le u \le 1, \quad 0 \le v \le 1 - u. \square$$

(b) Find the circulation of $\vec{F} = \langle x+y, y-z, x-y\rangle$ counterclockwise around the triangle (relative to the normal to the plane of the triangle having positive z-component).

Apply Stokes' theorem.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+z & y-z & x-y \end{vmatrix} = \langle 0,-1,-1\rangle.$$

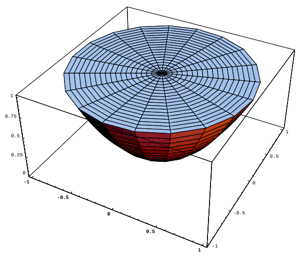
I will use the triangle itself — or rather the part of the plane containing it which is bounded by the triangle — as my surface. The normal is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \langle -2,-1,4\rangle.$$

(Note that this normal has positive z-component!)
Then $\operatorname{curl} \vec{F} \cdot \vec{N} = 0 + 1 - 4 = -3$. By Stokes' theorem, the circulation around the triangle is

$$\int \vec{F} \cdot \overrightarrow{ds} = \int_0^1 \int_0^{1-u} (-3)\, dv\, du = -\frac{3}{2}. \square$$

2. Find the flux of $\vec{F} = \langle x^3, y^3, z\rangle$ out through the boundary of the region $x^2 + y^2 \le z \le 1$.



Use the Divergence Theorem.

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 1.$$

Since $x^2 + y^2 = z = 1$ is the unit circle, the region projects to the unit disk $x^2 + y^2 \le 1$ in the x-y plane. I'll convert to cylindrical. The region is

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad r^2 \le z \le 1.$$

The divergence is $\operatorname{div} \vec{F} = 3r^2 + 1$. By the Divergence Theorem, the flux is

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^1 (3r^2 + 1)\, dz\, r\, dr\, d\theta = \pi. \ \square$$

3. Solve the one-dimensional wave equation

$$u_{tt} = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0$$

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x \le 2 \end{cases}$$

$$u_t(x,0) = -2, \qquad 0 \le x \le 2$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{2} \left(b_n \cos(\pi n t) + k_n \sin(\pi n t)\right).$$

I need to determine the coefficients.

b_n is the n-th Fourier sine coefficient of the initial displacement:

$$b_n = \frac{2}{2} \int_0^2 u(x,0)\, dx = \int_0^1 x^2 \sin \frac{\pi n x}{2}\, dx + \int_1^2 (2 - x) \sin \frac{\pi n x}{2}\, dx =$$

$$\left[-\frac{2}{\pi n} x^2 \cos \frac{\pi n x}{2} + \frac{8}{\pi^2 n^2} x \sin \frac{\pi n x}{2} + \frac{16}{\pi^3 n^3} \cos \frac{\pi n x}{2}\right]_0^1 + \left[-\frac{2}{\pi n}(2 - x) \cos \frac{\pi n x}{2} - \frac{4}{\pi^2 n^2} \sin \frac{\pi n x}{2}\right]_1^2 =$$

$$\frac{12}{\pi^2 n^2} \sin \frac{\pi n}{2} + \frac{16}{\pi^3 n^3} \left(\cos \frac{\pi n}{2} - 1\right).$$

Next, $(\pi n) k_n$ is the n-th Fourier sine coefficient of the initial velocity. Why? Differentiate the series for $u(x,t)$ with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{2} \left(-(\pi n) b_n \sin(\pi n t) + (\pi n) k_n \cos(\pi n t)\right).$$

Then

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{2} \cdot (\pi n) k_n \cos(\pi n t).$$

This says that $(\pi n) k_n$ is the n-th Fourier sine coefficient of $u_t(x,0)$.

Here, then, is the computation for k_n:

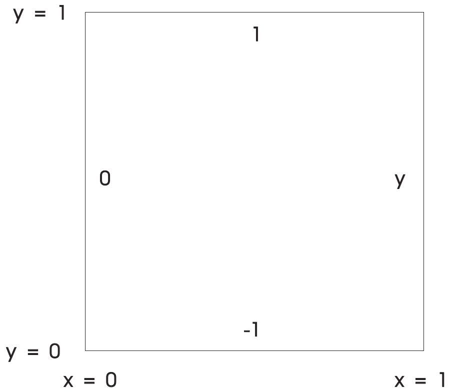
$$(\pi n) k_n = \frac{2}{2} \int_0^2 (-2) \sin \frac{\pi n x}{2}\, dx = \left[\frac{4}{\pi n} \cos \frac{\pi n x}{2}\right]_0^2 = \frac{4}{\pi n}(\cos(\pi n) - 1).$$

Hence, $k_n = \dfrac{4}{\pi^2 n^2}(\cos(\pi n) - 1)$.
The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{2} \left[\left(\frac{12}{\pi^2 n^2} \sin \frac{\pi n}{2} + \frac{16}{\pi^3 n^3} \left(\cos \frac{\pi n}{2} - 1 \right) \right) \cos(\pi n t) + \frac{4}{\pi^2 n^2} (\cos(\pi n) - 1) \sin(\pi n t) \right]. \square$$

4. Solve Laplace's equation $u_{xx} + u_{yy} = 0$ on the square $0 \le x \le 1$, $0 \le y \le 1$ with the boundary conditions

$$u(0,y) = 0, \quad u(1,y) = y, \quad u(x,0) = -1, \quad u(x,1) = 1.$$



Here are the four series for the top, right, bottom, and left solutions:

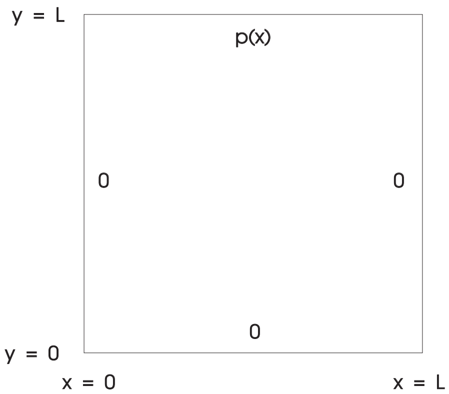
$$\sum_{n=1}^{\infty} k_n \sin \frac{\pi n x}{L} \sinh \frac{\pi n y}{L}$$

$$\sum_{n=1}^{\infty} k_n \sinh \frac{\pi n x}{L} \sin \frac{\pi n y}{L}$$

$$\sum_{n=1}^{\infty} k_n \sin \frac{\pi n x}{L} \sinh \frac{\pi n (L-y)}{L}$$

$$\sum_{n=1}^{\infty} k_n \sinh \frac{\pi n (L-x)}{L} \sin \frac{\pi n y}{L}$$

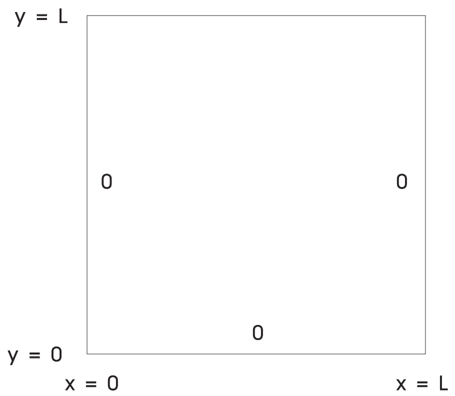
For example, the first series is the solution to a Dirichlet problem that has a function specified on the *top* of the square and $u = 0$ on the sides and bottom:



There is an easy way to check that you have the right series. Simply plug the sides of the square $x = 0$, $x = L$, $y = 0$, $y = L$ into the series and see which sides make the series 0. For example, consider the series

$$\sum_{n=1}^{\infty} k_n \sin \frac{\pi n x}{L} \sinh \frac{\pi n y}{L}$$

Setting $x = 0$, $x = L$, or $y = 0$ makes this series 0. That is, the series is giving these boundary values:



Since the series is 0 on the sides and bottom, it must be the series for the top problem.

In the given situation, I need to solve the top problem with $u = 1$ on the top, the bottom problem with -1 on the bottom, and the right problem with y on the right. Then I add these three solutions to obtain the solution to the original Dirichlet problem.

First, I'll do the top problem. $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u = 1$:

$$k_n \sinh(\pi n) = \frac{2}{1} \int_0^1 \sin(\pi n x)\, dx = \frac{2}{\pi n}(1 - \cos(\pi n)).$$

Hence, $k_n = \dfrac{2}{\pi n} \dfrac{1 - \cos(\pi n)}{\sinh(\pi n)}$. The solution to the top problem is

$$\sum_{n=1}^{\infty} \frac{2}{\pi n} \frac{1 - \cos(\pi n)}{\sinh(\pi n)} \sin(\pi n x) \sinh(\pi n y).$$

For the bottom problem, $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u = -1$. By the computation I just did, $k_n = \dfrac{2}{\pi n} \dfrac{\cos(\pi n) - 1}{\sinh(\pi n)}$. The solution to the bottom problem is

$$\sum_{n=1}^{\infty} \frac{2}{\pi n} \frac{\cos(\pi n) - 1}{\sinh(\pi n)} \sin(\pi n x) \sinh(\pi n(1 - y)).$$

Finally, for the right problem $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u = y$:

$$k_n \sinh(\pi n) = \frac{2}{1} \int_0^1 y \sin(\pi n y)\, dy = 2 \left[-\frac{1}{\pi n} y \cos(\pi n y) + \frac{1}{\pi^2 n^2} \sin(\pi n y) \right]_0^1 = -\frac{2}{\pi n} \cos(\pi n).$$

Therefore, $k_n = -\dfrac{2}{\pi n} \dfrac{\cos(\pi n)}{\sinh(\pi n)}$. The solution to the right problem is

$$\sum_{n=1}^{\infty} -\frac{2}{\pi n} \frac{\cos(\pi n)}{\sinh(\pi n)} \sinh(\pi n x) \sin(\pi n y).$$

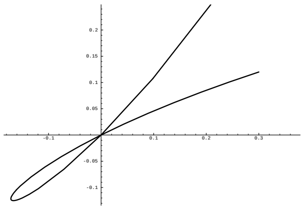
The solution to the original problem is

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{\pi n}\frac{1-\cos(\pi n)}{\sinh(\pi n)}\sin(\pi nx)\sinh(\pi ny)+\sum_{n=1}^{\infty}\frac{2}{\pi n}\frac{\cos(\pi n)-1}{\sinh(\pi n)}\sin(\pi nx)\sinh(\pi n(1-y))+$$

$$\sum_{n=1}^{\infty}-\frac{2}{\pi n}\frac{\cos(\pi n)}{\sinh(\pi n)}\sinh(\pi nx)\sin(\pi ny).\ \square$$

5. Find the area of the loop of

$$x=\frac{(t-1)(t-2)}{t},\quad y=\frac{(t-1)(t-2)}{t^2}.$$



$x=0$ and $y=0$ for $t=1$ and $t=2$. The loop is traced out as t goes from 1 to 2.

By Green's theorem, the area is given by

$$\int_{\vec{\sigma}} x\,dy.$$

Since $y=1-\frac{3}{t}+\frac{2}{t^2}$,

$$\frac{dy}{dt}=\frac{3}{t^2}-\frac{4}{t^3}.$$

Hence,

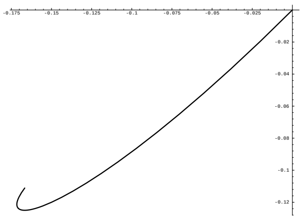
$$x\frac{dy}{dt}=-\frac{8}{t^4}+\frac{18}{t^3}-\frac{13}{t^2}+\frac{3}{t}.$$

Then

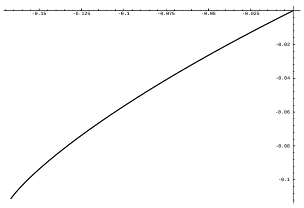
$$\int_1^2\left(-\frac{8}{t^4}+\frac{18}{t^3}-\frac{13}{t^2}+\frac{3}{t}\right)dt=3\ln 2-\frac{25}{12}\approx -0.00389.$$

Huh?

The problem is that Green's theorem requires that the curve be traversed *counterclockwise* around the region it encloses. Here is the part of the curve from 1 to 1.5:



Here is the part of the curve from 1.5 to 2:



From the pictures, it's clear that as t goes from 1 to 2 the loop is traced out *clockwise*. Hence, the integral above is the *negative* of the area; the area is $\dfrac{25}{12} - 3\ln 2 \approx 0.00389$. $\square$