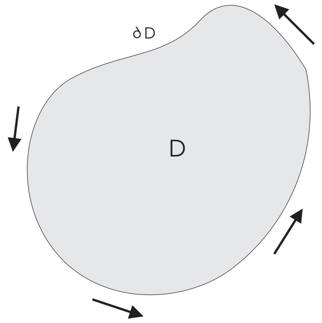
Green's theorem.

Let D be a "nice" region in the plane (where "'nice" means the boundary ∂D has a continuous parametrization and does not intersect itself). The boundary should be traversed in the *counterclockwise* direction. Suppose that $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ is a vector field, and P and Q have continuous partial derivatives. **Green's theorem** relates the line integral around the boundary ∂D to the double integral over D:

$$\int_{\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy.$$

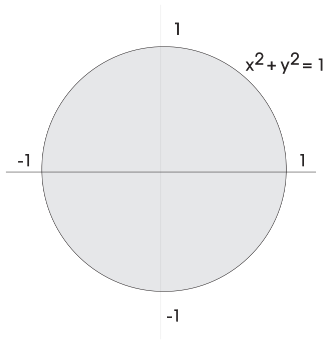


Green's theorem is a special case of **Stokes' theorem**; to peek ahead a bit, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is just the z component of the curl of $\vec{F}$, where $\vec{F}$ is regarded as a 3-dimensional vector field with zero z component:

$$\operatorname{curl}\langle P(x,y), Q(x,y), 0\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle.$$

Example. Let D be the unit disk $x^2 + y^2 \le 1$. Its boundary ∂D is the unit circle $x^2 + y^2 = 1$, which I may parametrize by

$$x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi.$$



I'll verify that Green's theorem holds for the line integral

$$\int_{\partial D} y^3\,dx - x^3\,dy.$$

First, I'll compute the line integral directly.

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.$$

So

$$\int_{\partial D} y^3\,dx - x^3\,dy = \int_0^{2\pi} \left((\sin t)^3(-\sin t) - (\cos t)^3(\cos t)\right)\,dt = \int_0^{2\pi} \left(-(\cos t)^4 - (\sin t)^4\right)\,dt.$$

Apply the double angle formulas:

$$-(\cos t)^4 - (\sin t)^4 = -\frac{1}{4}\left((1+\cos 2t)^2 + (1-\cos 2t)^2\right) = -\frac{1}{4}\left(2 + 2(\cos 2t)^2\right) = -\frac{1}{2}\left(1 + \frac{1}{2}(1+\cos 4t)\right).$$

So I have

$$\int_0^{2\pi} -\frac{1}{2}\left(1 + \frac{1}{2}(1+\cos 4t)\right)\,dt = -\frac{3\pi}{2}.$$

On the other hand, Green's theorem says the line integral is equal to

$$\int\!\!\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy = \int\!\!\int_{x^2+y^2\le 1} \left(-3x^2 - 3y^2\right)\,dx\,dy.$$

Convert to polar. The region is

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

So the integral becomes

$$\int_0^1 \int_0^{2\pi} -3r^2 \cdot r\,d\theta\,dr = -\frac{3\pi}{2}.$$

The results agree. □

Example. Verify Green's theorem for $\vec{F} = \langle x - y, x + y\rangle$, $\vec{\sigma}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$.

The curve is the unit circle again, and the region D it encloses is the disk $x^2 + y^2 \le 1$.

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t.$$

So the line integral is

$$\int_{\vec{\sigma}} (x-y)\,dx + (x+y)\,dy = \int_0^{2\pi} \left((\cos t - \sin t)(-\sin t) + (\cos t + \sin t)(\cos t)\right)\,dt = \int_0^{2\pi} dt = 2\pi.$$

Green's theorem says that the line integral is equal to the double integral

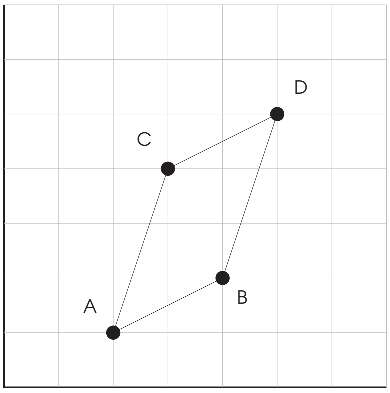
$$\int\!\!\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy = \int\!\!\int_{x^2+y^2\le 1} (1 - (-1))\,dx\,dy = 2\cdot(\text{the area of the circle}) = 2\pi.$$

The results agree. □

Example. Compute

$$\int_{\partial P} (-2y + 3x^2y + xy^2)\,dx + (x^2y + x^3 + 3x)\,dy,$$

where P is the parallelogram with vertices $A(2,1)$, $B(4,2)$, $C(3,4)$, and $D(5,5)$, and the boundary is traversed counterclockwise.



To do the line integral directly, I'd need to parametrize each side and compute the integral over each side. Rather than do four integrals, I'll use Green's theorem:

$$\int_{\partial P} (-2y + 3x^2y + xy^2)\,dx + (x^2y + x^3 + 3x)\,dy = \int\int_P \left((2xy + 3x^2 + 3) - (-2 + 3x^2 + 2xy)\right)\,dx\,dy =$$

$$\int\int_P 5\,dx\,dy = 5 \cdot (\text{the area of } P).$$

I don't need to compute the double integral; the area of a parallelogram is the length of the cross product of the vectors for two adjacent sides. $\overrightarrow{AB} = \langle 2, 1, 0\rangle$ and $\overrightarrow{AC} = \langle 1, 3, 0\rangle$, so

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \langle 0, 0, 5\rangle.$$

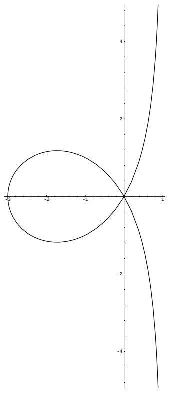
Now $|\overrightarrow{AB} \times \overrightarrow{AC}| = 5$, so the integral is $5 \cdot 5 = 25$. $\square$

In the last two examples, the double integral reduced to a number times the area of the region. You can use Green's theorem to find the area of a region D as follows. Compute the line integral $\int_{\partial D} x\,dy$. By Green's theorem,

$$\int_{\partial D} x\,dy = \int\int_D (1 - 0)\,dx\,dy = \text{the area of } D.$$

Example. The **trisectrix of MacLaurin** is given by the parametric equations

$$x = 1 - 4(\cos t)^2, \quad y = (\tan t)\left(1 - 4(\cos t)^2\right).$$



(The trisectrix is the pedal curve of a parabola; the pedal point is the reflection of the focus across the directrix.)

The entire curve is traced out from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. The loop is traced out from $-\dfrac{\pi}{3}$ to $\dfrac{\pi}{3}$. I'll find the area of the region enclosed by the loop.

$$x\frac{dy}{dt} = \left(1 - 4(\cos t)^2\right)\left[(\sec t)^2\left(1 - 4(\cos t)^2\right) + (\tan t)(8)(\sin t)(\cos t)\right] =$$

$$\left(1 - 4(\cos t)^2\right)\left((\sec t)^2 - 4 + 8(\sin t)^2\right) = (\sec t)^2 + 8(\cos t)^2 - 32(\sin t)^2(\cos t)^2.$$

The area is

$$\int_{-\pi/3}^{\pi/3} \left[(\sec t)^2 + 8(\cos t)^2 - 32(\sin t)^2(\cos t)^2\right]\, dt = 3\sqrt{3} \approx 5.19615.$$

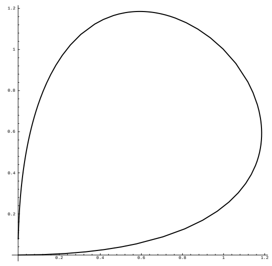
(You can integrate the second and third terms using the double angle formulas for sine and cosine.) □

Example. (a) Parametrize $(x+y)^3 = xy$. (Hint: Let $y = xt$.)

Following the hint, set $y = xt$. Then

$$(x+xt)^3 = 8x^2 t, \quad x^3(1+t)^3 = 8x^2 t, \quad x = \frac{8t}{(1+t)^3}.$$

Hence, $y = xt = \dfrac{8t^2}{(1+t)^3}$.



$y = xt$ is a line through the origin with slope t. Thus, the curve is being parametrized by the slope of the line joining the origin to a point on the curve. □

(b) Find the area enclosed by the loop of the curve.

The loop is traced as t goes from 0 to ∞.

$$\frac{dy}{dt} = \frac{8t(2-t)}{(1+t)^4},$$

so

$$x\frac{dy}{dt} = \frac{8t}{(1+t)^3} \cdot \frac{8t(2-t)}{(1+t)^4}.$$

The area is

$$\int_0^\infty \frac{8t}{(1+t)^3} \cdot \frac{8t(2-t)}{(1+t)^4}\, dt.$$

You can do this integral by letting $u = 1+t$, so $du = dt$. As t goes from 0 to ∞, u goes from 1 to ∞. The integral becomes

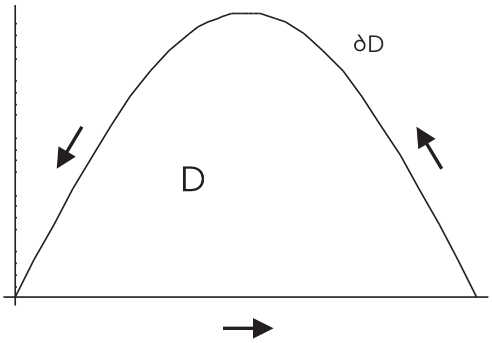
$$\int_1^\infty \frac{64(u-1)^2(3-u)}{u^7}\, du = \frac{16}{15}.$$

(Multiply out the u stuff on top, then divide each term by u^7.) □

Example. Let D be the region bounded by $y = \sin x$, from $x = 0$ to $x = \pi$, and the x-axis. Compute

$$\int_{\partial D} (x+y)^2\, dx - (x-y)^2\, dy,$$

assuming that the boundary is traversed counterclockwise.



The region is

$$0 \le x \le \pi, \quad 0 \le y \le \sin x.$$

By Green's theorem,

$$\int_{\partial D} (x+y)^2\,dx - (x-y)^2\,dy = \int_0^\pi \int_0^{\sin x} \left(-2(x-y) - 2(x+y)\right)\,dy\,dx = \int_0^\pi \int_0^{\sin x} -4x\,dy\,dx =$$

$$\int_0^\pi -4x\sin x\,dx = -4\pi.$$

If you compute the line integral directly, you need to parametrize the segment which makes up the base of the region and the curve. However, the curve is $y = \sin x$ as x goes from π to 0, because the boundary of the region is traversed counterclockwise. □