11-30-1993

Laplace's equation.

The **Laplacian** of $u(x, y, \ldots)$ is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots = u_{xx} + u_{yy} + \cdots.$$

Laplace's equation is $\nabla^2 u = 0$, or

$$u_{xx} + u_{yy} + \cdots = 0.$$

A function which satisfies Laplace's equation is **harmonic**.

Laplace's equation arises in many physical situations. For example, consider the two-dimensional heat equation $u_t = a^2(u_{xx} + u_{yy})$. The **steady state** temperature distribution — that is, the temperature distribution "after a lot of time has passed" — does not change with time, so $u_t = 0$. This means that $u_{xx} + u_{yy} = 0$, which is Laplace's equation.

You can solve Laplace's equation using separation of variables. For simplicity, I'll look at the two-dimensional case: $u_{xx} + u_{yy} = 0$. Set $u(x, y) = f(x)g(y)$. Then

$$f''(x)g(y) + f(x)g''(y) = 0, \quad \text{or} \quad \frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)}.$$

The left side depends only on x, while the right side depends only on y, so both sides are constant.

If the constant is 0, I get

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = 0.$$

Then $f''(x) = 0$, and integrating twice yields $f(x) = ax + b$. Likewise, $g''(y) = 0$, so $g(y) = cy + d$. The solution is

$$u(x, y) = (ax + b)(cx + d), \quad \text{where} \quad a, b, c, d \text{ are constants.}$$

Suppose that the constant is nonzero. I'll assume the constant is negative ($-\lambda^2$); by symmetry, making the constant positive (λ^2) merely swaps the roles of x and y.

The equation is

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = -\lambda^2.$$

First, $f''(x) = -\lambda^2 f(x)$, so

$$f(x) = c_1 \cos \lambda x + c_2 \sin \lambda x.$$

Next, $g''(y) = \lambda^2 g(y)$, so

$$g(y) = d_1 e^{\lambda y} + d_2 e^{-\lambda y}.$$

To proceed further, I need more information about u, and about the region where u is defined. In the two-dimensional case under consideration, the region will be a region in the plane — for example, a square.

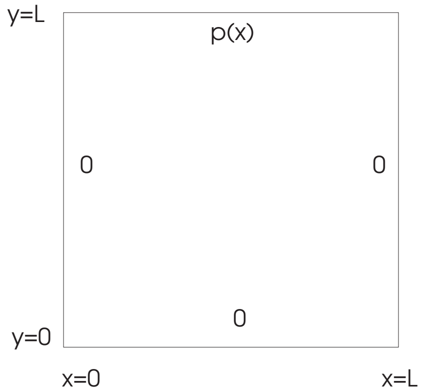
It's common to provide more information about u *by saying how u behaves on the boundary of the region.* Two common cases:

1. You can specify the values of u on the boundary. This is called a **Dirichlet problem**.

2. You can specify the values of u_x or u_y on the boundary. This is called a **Neumann problem**.

The Dirichlet problem for a square.

Suppose u is defined on the square $0 \le x \le L$, $0 \le y \le L$. Consider the case where $u = 0$ on the left, right, and bottom sides and $u(x, y) = p(x)$ on the top.



I'll often draw pictures like the one above to indicate the boundary conditions for a Laplace problem. In equation form, the boundary conditions are

$$u(0, y) = u(L, y) = u(x, 0) = 0, \quad u(x, L) = p(x), \qquad 0 \le x \le L, \quad 0 \le y \le L.$$

Now it's just a matter of plugging the boundary conditions into u. Recall that $u(x, y) = f(x)g(y)$, where

$$f(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \quad \text{and} \quad g(y) = d_1 e^{\lambda y} + d_2 e^{-\lambda y}.$$

First,

$$0 = u(0, y) = c_1 g(y), \quad \text{so} \quad c_1 = 0.$$

(If $g(y) = 0$, then $u \equiv 0$ — the trivial solution, and not a very interesting one!)
Now $f(x) = c_2 \sin \lambda x$. Next,

$$0 = u(L, y) = f(L)g(y), \quad f(L) = 0, \quad c_2 \sin \lambda L = 0.$$

If $c_2 = 0$, $f(x) = 0$, and $u \equiv 0$. So take $\sin \lambda L = 0$. This means $\lambda L = n\pi$ for some integer n, or $\lambda = \dfrac{\pi n}{L}$. Now

$$u(x, y) = \left(c_2 \sin \frac{\pi n x}{L}\right) \left(d_1 \exp\left(\frac{\pi n y}{L}\right) + d_2 \exp\left(-\frac{\pi n y}{L}\right)\right).$$

Now

$$0 = u(x, 0) = \left(c_2 \sin \frac{\pi n x}{L}\right)(d_1 + d_2), \quad \text{so} \quad d_1 + d_2 = 0.$$

Then $d_2 = -d_1$, so

$$u(x, y) = \left(c_2 \sin \frac{\pi n x}{L}\right) \left(d_1 \exp\left(\frac{\pi n y}{L}\right) - d_1 \exp\left(-\frac{\pi n y}{L}\right)\right) =$$

$$(2c_2 d_1)\left(\sin \frac{\pi n x}{L}\right) \cdot \frac{1}{2} \cdot \left(\exp\left(\frac{\pi n y}{L}\right) - \exp\left(-\frac{\pi n y}{L}\right)\right) = D \sin \frac{\pi n x}{L} \sinh \frac{\pi n y}{L}.$$

Superpose:

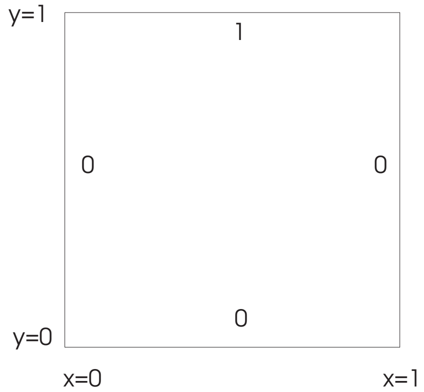
$$u(x, y) = \sum_{n=1}^{\infty} k_n \sin \frac{\pi n x}{L} \sinh \frac{\pi n y}{L}.$$

To identify the coefficient k_n, insert the remaining boundary condition:

$$p(x) = u(x, L) = \sum_{n=1}^{\infty} k_n \sin \frac{\pi n x}{L} \sinh(\pi n).$$

This equation says that $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u(x, L) = p(x)$.

Example. Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ with the following boundary conditions:



This is the square $0 \le x \le 1$, $0 \le y \le 1$. The boundary conditions are

$$u(0, y) = u(1, y) = u(x, 0) = 0, \quad u(x, 1) = 1, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

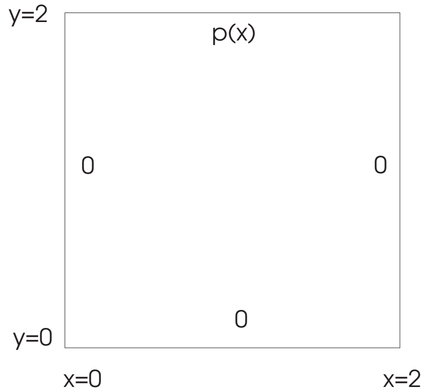
$k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u(x, L) = 1$:

$$k_n \sinh(\pi n) = \frac{2}{1} \int_0^1 (1) \sin(\pi n x)\, dx = \frac{2}{\pi n}(1 - \cos(\pi n)).$$

Hence, $k_n = \dfrac{2}{\pi n} \dfrac{1 - \cos(\pi n)}{\sinh(\pi n)}$, and the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \frac{1 - \cos(\pi n)}{\sinh(\pi n)} \sin(\pi n x) \sinh(\pi n y). \ \square$$

Example. Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ with the following boundary conditions:



where

$$p(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x \le 2 \end{cases}$$

$k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $u(x, L) = p(x)$:

$$k_n \sinh(\pi n) = \frac{2}{2}\left(\int_0^1 x \sin\frac{\pi n x}{2}\,dx + \int_1^2 (2-x)\sin\frac{\pi n x}{2}\,dx\right) =$$

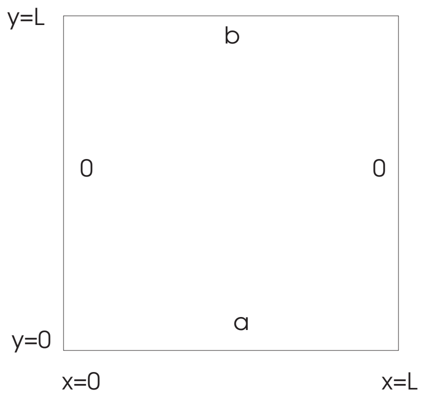
$$\left[-\frac{2}{\pi n}x\cos\frac{\pi n x}{2} + \frac{4}{\pi^2 n^2}\sin\frac{\pi n x}{2}\right]_0^1 + \left[-\frac{2}{\pi n}(2-x)\cos\frac{\pi n x}{2} - \frac{4}{\pi^2 n^2}\sin\frac{\pi n x}{2}\right]_1^2 = \frac{8}{\pi^2 n^2}\sin\frac{\pi n}{2}.$$

Therefore, $k_n = \dfrac{8}{\pi^2 n^2}\dfrac{\sin\frac{\pi n}{2}}{\sinh(\pi n)}$, and the solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2}\frac{\sin\frac{\pi n}{2}}{\sinh(\pi n)}\sin\frac{\pi n x}{2}\sinh\frac{\pi n y}{2}.\ \square$$

It's possible to use separation of variables to solve more complicated Dirichlet problems directly, though there is a method involving superposing solutions that is simpler in principle. I'll work out one example using separation, then explain the superposition idea.

Consider the Dirichlet problem



(It doesn't matter whether a and b are constants or functions.) If you go back to the separation of variables argument I did earlier, you will see that everything goes through as before up to the point where I knew that

$$u(x, y) = \left(c_2 \sin\frac{\pi n x}{L}\right)\left(d_1 \exp\left(\frac{\pi n y}{L}\right) + d_2 \exp\left(-\frac{\pi n y}{L}\right)\right).$$

By superposition, I get

$$u(x, y) = \sum_{n=1}^{\infty} \sin\frac{\pi n x}{L}\left(j_n \exp\left(\frac{\pi n y}{L}\right) + k_n \exp\left(-\frac{\pi n y}{L}\right)\right).$$

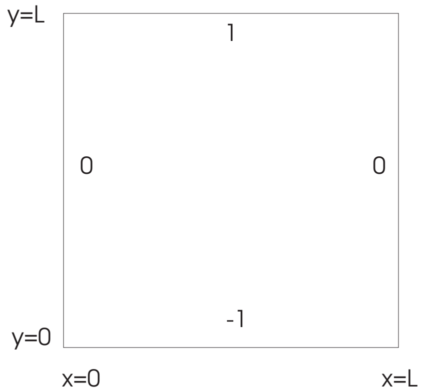
Now plug in the boundary conditions on the top and bottom:

$$a = u(x, 0) = \sum_{n=1}^{\infty}(j_n + k_n)\sin\frac{\pi n x}{L}$$

$$b = u(x, 0) = \sum_{n=1}^{\infty}(j_n e^{\pi n} + k_n e^{-\pi n})\sin\frac{\pi n x}{L}$$

This says that $j_n + k_n$ is the n-th Fourier sine coefficient of a and $j_n e^{\pi n} + k_n e^{-\pi n}$ is the n-th Fourier sine coefficient of b. Once you've computed the integrals for these coefficients, you'll have two equations to be solved for the unknowns j_n and k_n.

Example. Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ with the following boundary conditions:



$j_n + k_n$ is the n-th Fourier sine coefficient of $a = -1$:

$$j_n + k_n = \frac{2}{L}\int_0^L (-1)\sin\frac{\pi n x}{L}\,dx = \frac{2}{\pi n}(\cos(\pi n) - 1).$$

$j_n e^{\pi n} + k_n e^{-\pi n}$ is the n-th Fourier sine coefficient of $b = 1$:

$$j_n e^{\pi n} + k_n e^{-\pi n} = \frac{2}{L}\int_0^L (1)\sin\frac{\pi n x}{L}\,dx = \frac{2}{\pi n}(1 - \cos(\pi n)).$$

In matrix form, the two equations are

$$\begin{bmatrix} 1 & 1 \\ e^{\pi n} & e^{-\pi n} \end{bmatrix} \begin{bmatrix} j_n \\ k_n \end{bmatrix} = \frac{2}{\pi n}(\cos(\pi n) - 1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Use the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \frac{1}{AD - BC}\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}.$$

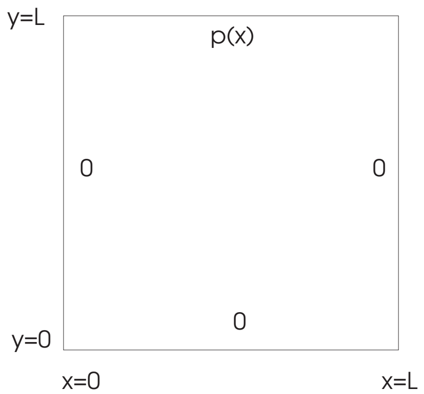
The solution is

$$\begin{bmatrix} j_n \\ k_n \end{bmatrix} = \frac{2}{\pi n}(\cos(\pi n) - 1)\cdot \frac{1}{e^{-\pi n} - e^{\pi n}} \begin{bmatrix} e^{-\pi n} & -1 \\ -e^{\pi n} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

To obtain the solution, find j_n and k_n and insert them into the series

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{\pi n x}{L}\left(j_n \exp\left(\frac{\pi n y}{L}\right) + k_n \exp\left(-\frac{\pi n y}{L}\right)\right). \square$$

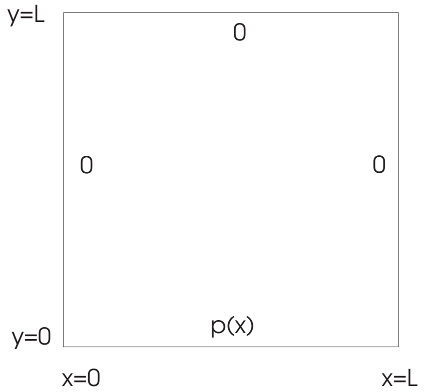
There is a way of thinking about this that is at least conceptually simpler. (You never avoid the ugly computations!) Return to the original Dirichlet problem $u_{xx} + u_{yy} = 0$ with the boundary conditions



It has the solution

$$u(x,y)=\sum_{n=1}^{\infty} k_n \sin\frac{\pi n x}{L}\sinh\frac{\pi n y}{L}.$$

Now suppose I want to solve the Dirichlet problem with boundary conditions



The idea is that the top and bottom are "swapped" by the substitution $y \to L-y$. So define

$$v(x,y)=u(x,L-y).$$

First, notice that $v(x,y)$ satisfies the following boundary conditions:

$$v(0,y)=u(0,L)=0, \quad v(L,y)=u(L,L-y)=0$$
$$v(x,0)=u(x,L)=p(x), \quad v(x,L)=u(x,0)=0$$

That is, v satisfies the boundary conditions with $p(x)$ on the bottom and 0 on the top, as I wanted. Also, v satisfies Laplace's equation:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2}{\partial x^2}u(x,L-y)+\frac{\partial^2}{\partial y^2}u(x,L-y)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Thus, the solution to this new Dirichlet problem is

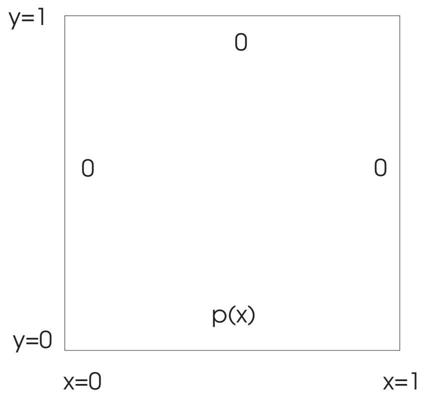
$$v(x,y)=u(x,L-y)=\sum_{n=1}^{\infty} k_n \sin\frac{\pi n x}{L}\sinh\frac{\pi n(L-y)}{L}.$$

Substitute the bottom boundary condition:

$$p(x)=v(x,0)=\sum_{n=1}^{\infty} k_n \sin\frac{\pi n x}{L}\sinh(\pi n).$$

Hence, $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $p(x)$.

Example. Solve the Dirichlet problem $u_{xx}+u_{yy}=0$ with the following boundary conditions:



where

$$p(x) = \begin{cases} x & \text{if } 0 \le x < \dfrac{1}{2} \\ 1 - x & \text{if } \dfrac{1}{2} \le x \le 2 \end{cases}$$

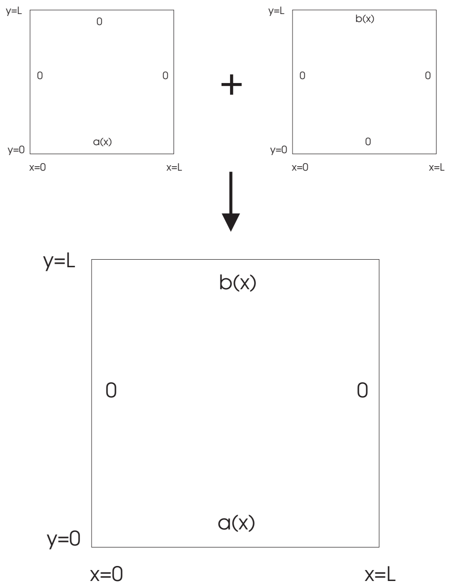
$k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient of $p(x)$:

$$k_n \sinh(\pi n) = \frac{2}{1}\left(\int_0^{1/2} x \sin(\pi n x)\, dx + \int_{1/2}^{1} (1-x)\sin(\pi n x)\, dx\right) = \frac{4}{\pi^2 n^2} \sin\frac{\pi n}{2}.$$

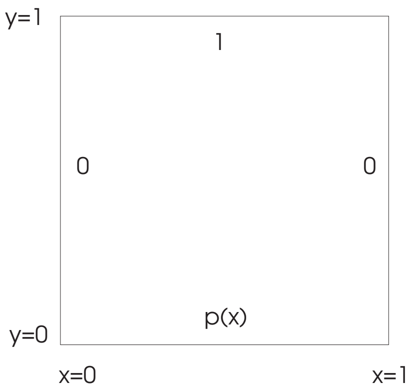
Hence, $k_n = \dfrac{4}{\pi^2 n^2} \dfrac{\sin \dfrac{\pi n}{2}}{\sinh \pi n}$, and the solution is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \sin\frac{\pi n}{2} \sin(\pi n x) \sinh\left[\pi n (1-y)\right]. \ \square$$

Now it's easy to see how to solve a Dirichlet problem with functions given on the top and bottom of the square. Just solve the "top problem" and the "bottom problem" and superpose!



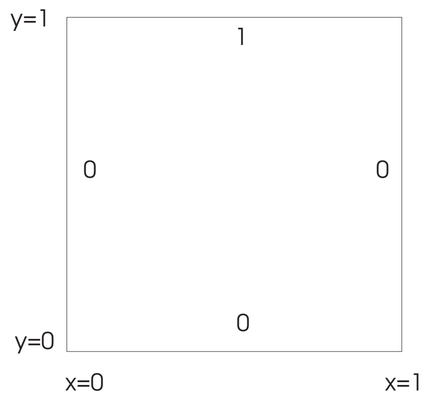
Example. Solve the Dirichlet problem $u_{xx} + u_{yy} = 0$ with the following boundary conditions:



where

$$p(x) = \begin{cases} x & \text{if } 0 \le x < \dfrac{1}{2} \\ 1 - x & \text{if } \dfrac{1}{2} \le x \le 2 \end{cases}$$

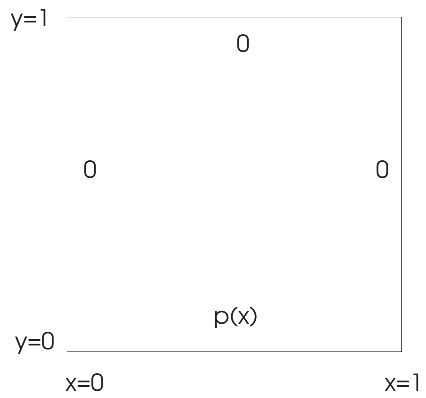
In an earlier example, I computed the solution to the Dirichlet problem



It was

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \frac{1 - \cos(\pi n)}{\sinh(\pi n)} \sin(\pi n x) \sinh(\pi n y).$$

In the preceding example, I computed the solution to the Dirichlet problem



It was

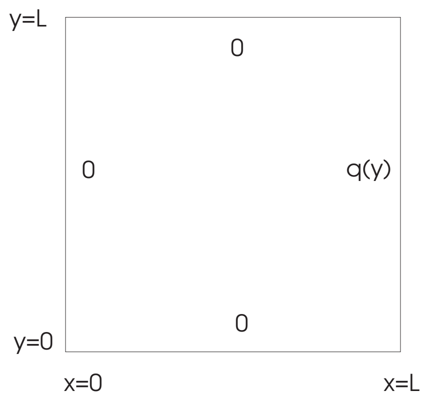
$$u(x,y) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \sin \frac{\pi n}{2} \sin(\pi n x) \sinh\left[\pi n (1 - y)\right].$$

The solution to the given Dirichlet problem is the sum of these two solutions:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \frac{1 - \cos(\pi n)}{\sinh(\pi n)} \sin(\pi n x) \sinh(\pi n y) + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \sin \frac{\pi n}{2} \sin(\pi n x) \sinh\left[\pi n (1 - y)\right]. \square$$

The obvious question is: What if you have functions given on the left or right sides? What if you have functions given on all four sides?

First, consider the Dirichlet problem

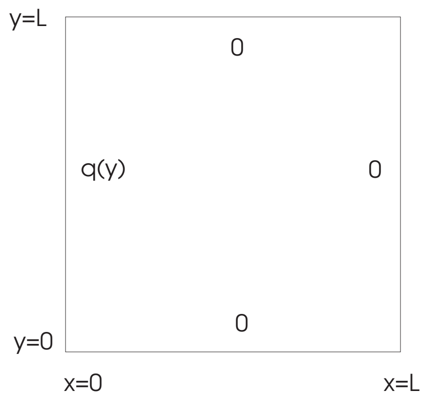


You can see this is the same as the "top" Dirichlet problem — the very first one I worked out — but with x and y switched. To get the solution, simply swap x and y in the solution to the "top" Dirichlet problem:

$$u(x,y) = \sum_{n=1}^{\infty} k_n \sinh \frac{\pi n x}{L} \sin \frac{\pi n y}{L}.$$

Here $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient for $q(y)$.

Consider the Dirichlet problem



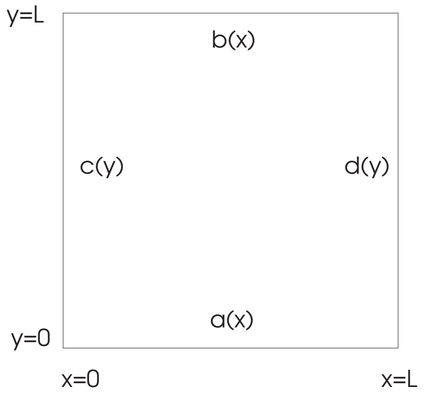
This is the same as the "bottom" Dirichlet problem but with x and y switched. To get the solution, simply swap x and y in the solution to the "bottom" Dirichlet problem:

$$u(x,y) = \sum_{n=1}^{\infty} k_n \sinh \frac{\pi n (L-x)}{L} \sin \frac{\pi n y}{L}.$$

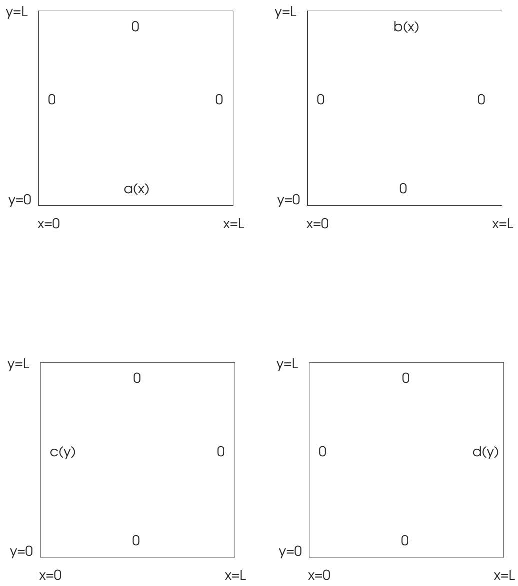
Again, $k_n \sinh(\pi n)$ is the n-th Fourier sine coefficient for $q(y)$.

I think you can guess what is coming next! I will spare you a horrible example and merely explain the idea.

To solve the Dirichlet problem



simply solve the following four Dirichlet problems:



The solution to the big problem is the sum of the four solutions to the little problems. The computations might be messy, but the idea is rather simple.