12-2-1993

Laplace's equation in polar coordinates.

In polar coordinates, the Laplacian has the form

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

In polar coordinates, Laplace's equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

I will solve the equation using separation of variables. As usual, I'll assume that the solution remains bounded for all r and θ.

Let $u(r,\theta) = f(r)g(\theta)$. Then

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta) = 0.$$

Multiply through by r^2, and divide through by $f(r)$ and by $g(\theta)$:

$$\frac{r^2 f''(r) + rf'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)}.$$

The left side depends only on r while the right side depends only on θ. Hence, both sides are constant:

$$\frac{r^2 f''(r) + rf'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = \lambda^2.$$

(If the constant is $-\lambda^2$, the g part becomes $-\dfrac{g''(\theta)}{g(\theta)} = -\lambda^2$, so $g(\theta) = c_1 e^{\lambda\theta} + c_2 e^{-\lambda\theta}$. But $g(\theta)$ should be periodic with period 2π. This will not be the case unless $c_1 = c_2 = 0$, which yields the trivial solution $u \equiv 0$. Hence, this case is ruled out.)

$-\dfrac{g''(\theta)}{g(\theta)} = \lambda^2$ gives

$$g(\theta) = c_1\cos(\lambda\theta) + c_2\sin(\lambda\theta).$$

$\dfrac{r^2 f''(r) + rf'(r)}{f(r)} = \lambda^2$ may be rearranged to obtain

$$r^2 f''(r) + rf'(r) - \lambda^2 f(r) = 0.$$

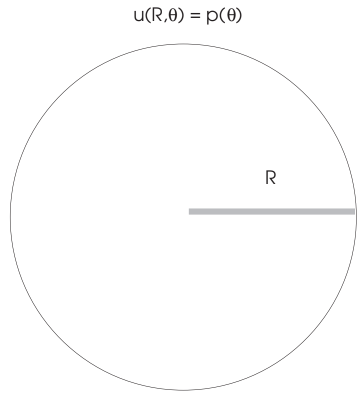
This is an **Euler equation**. It has solutions of the form r^k. To find the values of k which work, just plug in. If $f(r) = r^k$, then $f'(r) = kr^{k-1}$ and $f''(r) = k(k-1)r^{k-2}$. Hence,

$$r^2 \cdot k(k-1)r^{k-2} + r\cdot kr^{k-1} - \lambda^2 r^k = 0, \quad \text{or} \quad (k^2-\lambda^2)r^k = 0.$$

This is supposed to be an identity, so $k^2 - \lambda^2 = 0$, or $k = \pm\lambda$. So r^λ and $r^{-\lambda}$ work, and

$$f(r) = d_1 r^\lambda + d_2 r^{-\lambda}.$$

To go further, I'll need to consider a specific region. I will use the disk $0 \le r \le R$ of radius R centered at the origin. I'll look at the **Dirichlet problem** for the disk — that is, I'll assume that the values of u are specified on the boundary. This means that $u(R, \theta) = p(\theta)$ for some function $p(\theta)$.



Plug the boundary condition in:

$$p(\theta) = u(R, \theta) = f(R)g(\theta).$$

Now $p(\theta)$ must be periodic with period 2π: $p(\theta) = p(\theta + 2\pi)$ for all θ. So

$$f(R)g(\theta) = p(\theta) = p(\theta + 2\pi) = f(R)g(\theta + 2\pi), \quad \text{and} \quad g(\theta) = g(\theta + 2\pi).$$

This says that g is also periodic with period 2π. Now $g(\theta) = c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta)$, so

$$c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta) = c_1 \cos(\lambda(\theta + 2\pi)) + c_2 \sin(\lambda(\theta + 2\pi)).$$

If I set $\theta = -\pi$, the equation becomes

$$c_1 \cos(\lambda\pi) - c_2 \sin(\lambda\pi) = c_1 \cos(\lambda\pi) + c_2 \sin(\lambda\pi).$$

Hence, $c_2 \sin(\lambda\pi) = 0$. If $\sin(\lambda\pi) = 0$, then λ must be an integer: $\lambda = 0, 1, 2, \ldots$.

If $c_2 = 0$, then the equation above becomes $c_1 \cos(\lambda\theta) = c_1 \cos(\lambda(\theta + 2\pi))$. Setting $\theta = 0$ yields $c_1 = \cos(2\pi\lambda)$, and again λ must be an integer: $\lambda = 0, 1, 2, \ldots$.

So in any case, λ is an integer. I'l write $\lambda = n$, since n usually denotes an integer. In this case, $n = 0, 1, 2, \ldots$. Thus,

$$f(r) = d_1 r^n + d_2 r^{-n}, \quad g(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta).$$

Let $r \to 0+$ — that is, approach the center of the disk. Then $r^{-n} \to +\infty$, contradicting the assumption that u remains bounded. So there r^{-n} can't be there — $d_2 = 0$. This gives $f(r) = d_1 r^n$. At this point,

$$u(r, \theta) = r^n (A \cos(n\theta) + B \sin(n\theta).$$

Note that if $n = 0$ I get a constant solution. So superposition gives

$$u(r, \theta) = \frac{j_0}{2} + \sum_{n=1}^{\infty} r^n (j_n \cos(n\theta) + k_n \sin(n\theta).$$

(I've used $\dfrac{j_0}{2}$ as the constant term to make this look like a Fourier expansion.)

Plug in the boundary condition:

$$p(\theta) = u(R, \theta) = \frac{j_0}{2} + \sum_{n=1}^{\infty} R^n (j_n \cos(n\theta) + k_n \sin(n\theta).$$

Note that $\cos(n\theta)$, $\sin(n\theta)$ arise in the Fourier expansion of $p(\theta)$. The equation above says that $R^n j_n$ is the n-th Fourier cosine coefficient of $p(\theta)$ while $R^n k_n$ is the n-th Fourier sine coefficient of $p(\theta)$. Note that to get, e.g., $\cos(n\theta)$ from $\cos\dfrac{\pi n\theta}{L}$, L must be π. So it seems as though the correct expressions for the coefficient should be (for example)

$$R^n j_n = \frac{1}{\pi}\int_{-\pi}^{\pi} p(\theta)\cos(n\theta)\,d\theta.$$

However, as a practical matter the initial distribution $p(\theta)$ will often be given on $0 \le \theta \le 2\pi$ — with the understanding that you continue it periodically everywhere else. If this is the case and if you use the formula above, you will need to figure out what $p(\theta)$ should be for $-\pi \le \theta \le 0$.

For this reason, if $p(\theta)$ is given on $0 \le \theta \le 2\pi$, it is easier to use the following formulas for the coefficients:

$$R^n j_n = \frac{1}{\pi}\int_{0}^{2\pi} p(\theta)\cos(n\theta)\,d\theta.$$

$$R^n k_n = \frac{1}{\pi}\int_{0}^{2\pi} p(\theta)\sin(n\theta)\,d\theta.$$

To see that these work, note that $p(\theta)$, $\cos(n\theta)$, and $\sin(n\theta)$ all repeat every 2π units. So the integrals from $-\pi$ to 0 are the same as the integrals from π to 2π, and

$$\int_{-\pi}^{\pi}\text{junk} = \int_{-\pi}^{0}\text{junk} + \int_{0}^{\pi}\text{junk} = \int_{0}^{\pi}\text{junk} + \int_{\pi}^{2\pi}\text{junk} = \int_{0}^{2\pi}\text{junk}.$$

You can use the complex exponential as well:

$$R^n c_n = \frac{1}{\pi}\int_{0}^{2\pi} p(\theta)e^{ni\theta}\,d\theta.$$

Example. Solve the Dirichlet problem on the disk $0 \le r \le 1$ with the boundary condition

$$u(1,\theta) = p(\theta) = \begin{cases} \theta & \text{if } 0 \le \theta < \pi \\ 2\pi - \theta & \text{if } \pi \le \theta \le 2\pi \end{cases}, \quad \text{and} \quad p(\theta) = p(\theta + 2\pi) \text{ for all } \theta.$$

First,

$$j_0 = \frac{1}{\pi}\int_{0}^{2\pi} p(\theta)\,d\theta = \pi.$$

Next,

$$1^n c_n = \frac{1}{\pi}\int_{0}^{2\pi} p(\theta)e^{ni\theta}\,d\theta = \frac{1}{\pi}\left(\int_{0}^{\pi}\theta e^{ni\theta}\,d\theta + \int_{\pi}^{2\pi}(2\pi-\theta)e^{ni\theta}\,d\theta\right) =$$

$$\frac{1}{\pi}\left(\left[\left(-\frac{i\theta}{n} + \frac{1}{n^2}\right)e^{ni\theta}\right]_0^{\pi} + \left[\left(-\frac{i}{n}(2\pi-\theta) - \frac{1}{n^2}\right)e^{ni\theta}\right]_{\pi}^{2\pi}\right) =$$

$$\frac{2}{\pi n^2}(\cos(\pi n) - 1).$$

Thus, $j_n = \dfrac{2}{\pi n^2}(\cos(\pi n) - 1)$ and $k_n = 0$. The solution is

$$u(r,\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}(\cos(\pi n) - 1)\cdot r^n\cdot\cos(n\theta).\ \square$$