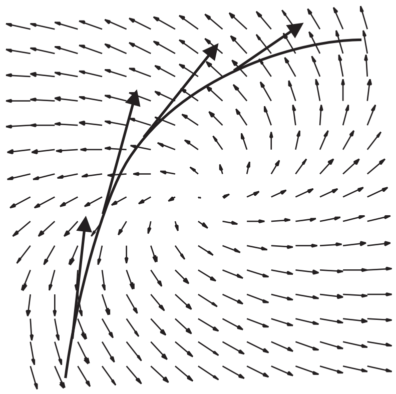
Line integrals.

If $\vec{F}$ is a vector field and $\vec{\sigma}(t)$, $a \le t \le b$, is a path, the **line integral** of $\vec{F}$ along $\vec{\sigma}(t)$ is

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_a^b \vec{F}(t) \cdot \vec{\sigma}\,'(t)\,dt.$$

Here is what this looks like in two dimensions.



The vector field looks like — well, a field of vectors filling up the plane. As you move along the path, at each point you compute the dot product of the velocity vector $\vec{\sigma}'(t)$ with the vector from the field at that point. Integrate to "add up" the results; the total is the line integral.

Notice that I'm writing $\overrightarrow{ds}$ instead of ds, the differential for a path integral. Here's the difference:

$$\overrightarrow{ds} = \vec{\sigma}\,'(t)\,dt, \quad \text{while} \quad ds = |\vec{\sigma}\,'(t)|\,dt.$$

One is a vector, the other is a scalar.

Example. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$ for $\vec{F}(x,y) = \langle y, x\rangle$ over the path $\vec{\sigma}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \dfrac{\pi}{2}$.

The path is simply the part of $x^2 + y^2 = 1$ lying in the first quadrant — a quarter circle. $\vec{\sigma}(0) = \langle 1, 0\rangle$ while $\vec{\sigma}\left(\dfrac{\pi}{2}\right) = \langle 0, 1\rangle$, so the quarter circle is traversed counterclockwise.

The velocity is

$$\vec{\sigma}\,'(t) = \langle -\sin t, \cos t\rangle.$$

Write the vector field in terms of t:

$$\vec{F}(t) = \langle \sin t, \cos t\rangle.$$

Then

$$\vec{F}(t) \cdot \vec{\sigma}\,'(t) = -(\sin t)^2 + (\cos t)^2 = \cos 2t.$$

(I used a double angle formula from trigonometry.) Therefore,

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_0^{\pi/2} \cos 2t\,dt = 0.$$

Keep $\vec{F}$ the same, but change the path to

$$\vec{\tau}(t) = \langle \cos 2t, \sin 2t\rangle, \quad 0 \le t \le \frac{\pi}{4}.$$

τ is the same quarter circle, but

$$\vec{\tau}\,'(t) = \langle -2\sin 2t, 2\cos 2t\rangle.$$

The extra factor of 2 means the path is traversed twice as fast as before. The field is

$$\vec{F}(t) = \langle \sin 2t, \cos 2t\rangle.$$

So

$$\vec{F}(t)\cdot\vec{\tau}\,'(t) = -2(\sin 2t)^2 + 2(\cos 2t)^2 = 2\cos 4t.$$

Therefore,

$$\int_{\vec{\tau}} \vec{F}\cdot\overrightarrow{ds} = \int_0^{\pi/4} 2\cos 4t\,dt = 0.$$

Traversing the path twice as rapidly made no difference. This is true in general: *If you traverse the same path in the same direction at different speeds, the line integral does not change.* □

Example. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F}\cdot\overrightarrow{ds}$ for $\vec{F}(x,y) = \langle x+y, x-y\rangle$ along the path $\vec{\sigma}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$.

The path is the part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

$$\vec{\sigma}\,'(t) = \langle 1, 2t\rangle \text{ and } \vec{F}(t) = \langle t+t^2, t-t^2\rangle.$$

Hence,

$$\vec{F}(t)\cdot\vec{\sigma}\,'(t) = t + 3t^2 - 2t^3.$$

The integral is

$$\int_{\vec{\sigma}} \vec{F}\cdot\overrightarrow{ds} = \int_0^1 (t + 3t^2 - 2t^3)\,dt = 1.$$

If I change the path to $\vec{\tau}(t) = \langle 1-t, (1-t)^2\rangle$, I am now traversing $y = x^2$ from $(1,1)$ to $(0,0)$. You can check that in this case,

$$\int_{\vec{\tau}} \vec{F}\cdot\overrightarrow{ds} = -1.$$

This is true in general: *If you traverse the same path in the opposite direction, the line integral is multiplied by -1.* □

Example. (Differential notation) If $\vec{F} = \langle F_1, F_2, F_3\rangle$ is a vector field and $\vec{\sigma}(t)$, $a \le t \le b$, is a path, then

$$\vec{F}\cdot\vec{\sigma}\,'(t) = \langle F_1, F_2, F_3\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}.$$

So

$$\int_{\vec{\sigma}} \vec{F}\cdot\overrightarrow{ds} = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt.$$

If I formally multiply the dt in, I get

$$\int_{\vec{\sigma}} \vec{F}\cdot\overrightarrow{ds} = \int_a^b (F_1\,dx + F_2\,dy + F_3\,dz).$$

You will sometimes see line integrals written in this form.

To do the computation, convert F_1, F_2, and F_3 to functions of t using $\vec{\sigma} = \langle x(t), y(t), z(t)\rangle$. Replace dx, dy, dz by $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, $\dfrac{dz}{dt}$, and integrate the whole thing with respect to t.

For example, I'll compute

$$\int_{\vec{\sigma}} x\,dx + y\,dy + (x + y - 1)\,dz,$$

where $\vec{\sigma}(t)$ is the segment joining $(1, 1, 1)$ to $(2, 3, 4)$.

The segment is

$$\vec{\sigma}(t) = (1 - t) \cdot (1, 1, 1) + t \cdot (2, 3, 4) = \langle 1 + t, 1 + 2t, 1 + 3t \rangle, \quad 0 \le t \le 1.$$

That is,

$$x = 1 + t, \quad y = 1 + 2t, \quad z = 1 + 3t.$$

Now

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 3.$$

Plugging everything into the integral, I get

$$\int_{\vec{\sigma}} x\,dx + y\,dy + (x + y - 1)\,dz = \int_0^1 [(1 + t)(1) + (1 + 2t)(2) + (1 + 3t)(3)]\ dt = 13. \square$$