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More on the heat equation.

A heat conduction problem has **homogeneous boundary conditions** if the ends of the rod or wire are held at a constant temperature of $0°$:

$$u(0,t) = u(L,t) = 0, \qquad t > 0.$$

Of course, there are many other possibilities. I'll look at two of them: The case where the ends are held at constant, *nonzero* temperatures and the case where the ends are insulated.

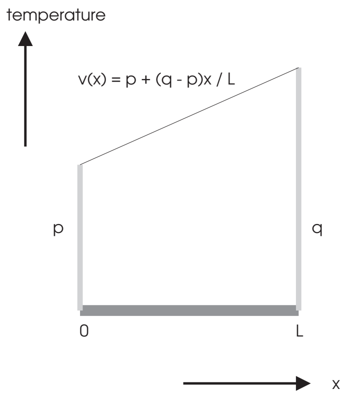
Suppose that the ends are held at constant temperatures $p°$ and $q°$:

$$u_t = a^2 u_{xx}, \qquad 0 < x < L, \quad t > 0$$

$$u(0,t) = p, \quad u(L,t) = q, \qquad t > 0$$

$$u(x,0) = f(x), \qquad 0 < x < L$$

As $t \to \infty$, the temperature in the rod approaches a **steady-state distribution** $v(x)$. It's intuitively plausible that this should simply be a linear distribution "connecting the endpoints" $(0,p)$ and (L,q).



Here is a mathematical verification.

Since $v(x)$ satisfies the heat equation, $0 = v_t = a^2 v_{xx}$, or $v_{xx} = 0$. Integrate twice with respect to x: $v(x) = cx + d$, for constants c and d. Plug in the points $(0,p)$ and (L,q):

$$v(x) = \frac{q-p}{L}x + p.$$

So now I know what the temperature distribution looks like after a lot of time has elapsed. I still need to find out what the temperature does for small values of t.

Write the general solution as a sum of the steady-state solution v and a **transient solution** $w(x,t)$:

$$u(x,t) = w(x,t) + v(x).$$

Then

$$w_t = \frac{\partial}{\partial t}(u(x,t) - v(x)) = u_t = a^2 u_{xx} = a^2 \frac{\partial^2}{\partial x^2}(w+v) = a^2 w_{xx} + a^2 \cdot 0 = a^2 w_{xx}.$$

Thus, w satisfies the heat equation.

Next,

$$w(0,t) = u(0,t) - v(0) = p - p = 0, \quad w(L,t) = u(L,t) - v(L) = q - q = 0.$$

Hence, w satisfies the homogeneous boundary conditions.
Finally,

$$f(x) = u(x,0) = w(x,0) + v(x), \text{ so } w(x,0) = f(x) - v(x).$$

It follows that w is a solution to the homogeneous boundary condition problem with the initial distribution $w(x,0) = f(x) - v(x)$. Therefore, by our solution to the homogeneous boundary condition problem,

$$w(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\pi^2 n^2 a^2 t}{L^2}\right) \sin\left(\frac{\pi n x}{L}\right).$$

Here b_n is the n-th Fourier sine coefficient of $w(x,0) = f(x) - v(x)$.

Example. Solve the heat conduction problem

$$u_t = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0$$

$$u(0,t) = 3, \quad u(2,t) = 7, \qquad t > 0$$

$$u(x,0) = 5 - 2x, \qquad 0 < x < 2$$

The steady-state solution is the line joining $(0,3)$, $(2,7)$:

$$v(x) = 2x + 3.$$

Write $u(x,t) = w(x,t) + v(x)$. Then

$$w(x,0) = u(x,0) - v(x) = (5 - 2x) - (2x + 3) = 2 - 4x.$$

Find the Fourier sine series for $2 - 4x$:

$$b_n = \frac{2}{2}\int_0^2 (2 - 4x)\sin\left(\frac{\pi n x}{2}\right)\,dx = \left[-\frac{2}{\pi n}(2 - 4x)\cos\left(\frac{\pi n x}{2}\right) - \frac{16}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)\right]_0^2 =$$

$$\left(\frac{12}{\pi n}\cos(\pi n) - 0\right) - \left(-\frac{4}{\pi n} - 0\right) = \frac{4}{\pi n}\left(1 - 3\cdot(-1)^n\right).$$

Therefore,

$$w(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n}\left(1 - 3\cdot(-1)^n\right)\exp(-\pi^2 n^2 t)\sin\left(\frac{\pi n x}{2}\right).$$

The solution is

$$u(x,t) = 2x + 3 + w(x,t).$$

$u(x,t)$ is a function of two variables, so I can use *Mathematica* to draw the graph.
First, I'll define a function that gives $(-1)^n$:

```
s[n_] := 1 /; Mod[n,2]==0

s[n_] := -1 /; Mod[n,2]==1
```

For the transient solution, I'll use the first ten terms of the series above.

```
w[x_, t_] := N[Sum[4/(Pi n) (1 - 3 s[n]) E^(-Pi^2 n^2 t) *
    Sin[Pi n x/2], {n, 1, 10}]]
```

Finally, here is the solution:

```
u[x_, t_] := 2x + 3 + w[x,t]
```

The negative exponential terms go to 0 quickly. I've drawn $u(x,t)$ in the first 0.05 time units. Notice how the initial wiggles have almost disappeared.

```
Plot3D[u[x,t], {x, 0, 2}, {t, 0, 0.005}, ViewPoint->{1.3,2.4,2}]
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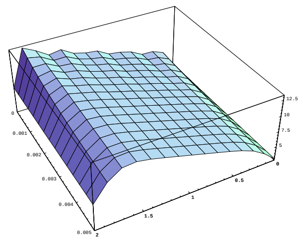

Next, here is the picture for $t = 0$ to $t = 2$. Notice that the transient part of the solution is almost dead at $t = 2$ (the front of the picture) — the solution has flattened out to a straight line, the steady state distribution $v(x)$.

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Plot3D[u[x,t], {x, 0, 2}, {t, 0, 2}, ViewPoint->{1.3,2.4,2}]
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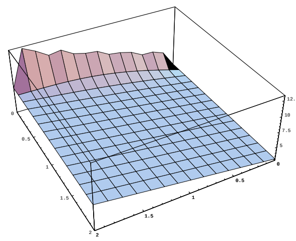

$\square$

Next, consider the case where the ends of the bar are *insulated*. (I always assume that the *sides* are insulated.) This means that the *rate* of heat transfer through the ends is 0:

$$u_x(0,t) = u_x(L,t) = 0, \qquad t > 0.$$

This heat conduction problem is described by the equations

$$u_t = a^2 u_{xx}, \qquad 0 < x < L, \quad t > 0$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \qquad 0 < x < L$$

Assume also that the solution is bounded: $|u(x,t)| \le M$ for some M.

I'll use separation of variables. The first part of the solution process is a rerun of the solution process for the homogeneous boundary condition case. I'll go through the steps for the sake of completeness.

Let $u(x,t) = g(x)h(t)$. Then

$$g(x)h'(t) = a^2 g''(x)h(t), \quad \frac{h'(t)}{h(t)} = a^2 \frac{g''(x)}{g(x)}.$$

The left side is a function of t while the right side is a function of x. Hence, both sides are constant:

$$\frac{h'(t)}{h(t)} = a^2 \frac{g''(x)}{g(x)} = -\lambda^2.$$

Solve the t equation:

$$\frac{h'(t)}{h(t)} = -\lambda^2, \quad \ln|h(t)| = -\lambda^2 t + c, h(t) = c_0 \exp(-\lambda^2 t).$$

The boundedness assumption shows that $-\lambda^2$ really is negative, and I may take λ to be a positive *real* constant.

Solve the x equation:

$$a^2 \frac{g''(x)}{g(x)} = -\lambda^2, \quad g''(x) + \left(\frac{\lambda}{a}\right)^2 g(x) = 0, \quad g(x) = c_1 \cos \frac{\lambda}{a} x + c_2 \sin \frac{\lambda}{a} x.$$

Therefore,

$$u(x,t) = \exp(-\lambda^2 t)\left(d_1 \cos \frac{\lambda}{a} x + d_2 \sin \frac{\lambda}{a} x\right).$$

Next, I'll plug in the boundary conditions. Differentiate:

$$u_x = \exp(-\lambda^2 t)\left(\frac{\lambda}{a}\right)\left(-d_1 \sin \frac{\lambda}{a} x + d_2 \cos \frac{\lambda}{a} x\right).$$

Then

$$0 = u_x(0,t) = \exp(-\lambda^2 t)\left(\frac{\lambda}{a}\right) \cdot d_2, \quad d_2 = 0.$$

Now

$$u_x = -d_1 \sin \exp(-\lambda^2 t)\left(\frac{\lambda}{a}\right) \sin \frac{\lambda}{a} x.$$

Plug in again:

$$0 = u_x(L,t) = -d_1 \sin \exp(-\lambda^2 t)\left(\frac{\lambda}{a}\right) \sin \frac{\lambda L}{a}.$$

$d_1 = 0$ gives the trivial solution $u(x,t) \equiv 0$. So assume $d_1 \neq 0$. Then $\dfrac{\lambda L}{a} = n\pi$, so $\lambda = \dfrac{\pi a n}{L}$.
So

$$u(x,t) = d_1 \exp\left(-\frac{\pi^2 a^2 n^2 t}{L^2}\right) \cos \frac{\pi n x}{L}.$$

Superpose:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\pi^2 a^2 n^2 t}{L^2}\right) \cos \frac{\pi n x}{L}.$$

Finally, plug in the initial condition:

$$f(x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{L}.$$

That is, the a_n are the Fourier cosine coefficients of the initial distribution $f(x)$.

In other words, when the ends are held at $0°$ ($u = 0$), the solution is obtained by computing a Fourier *sine* series. When the ends are insulated ($u_x = 0$), the solution is obtained by computing a Fourier *cosine* series.

Example. Solve the heat conduction problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0$$

$$u(x,0) = x + 1, \qquad 0 < x < 1$$

First,

$$a_0 = \frac{2}{1}\int_0^1 (x+1)\,dx = 2\left[\frac{1}{2}x^2 + x\right]_0^1 = 3.$$

Next,

$$a_n = \frac{2}{1}\int_0^1 (x+1)\cos(\pi n x)\,dx = 2\left[\frac{1}{\pi n}(x+1)\sin(\pi n x) + \frac{1}{\pi^2 n^2}\cos(\pi n x)\right]_0^1 = -\frac{2}{\pi^2 n^2}\left(1 - (-1)^n\right).$$

Therefore,

$$u(x,t) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}\left(1 - (-1)^n\right)\exp(-\pi^2 n^2 t)\cos(\pi n x). \ \square$$

Example. In this extended example, I'll review the different combinations of initial conditions and boundary conditions. Consider the heat equation

$$u_t = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0.$$

a. (Homogeneous boundary conditions.) If the initial distribution is a sum of sines *of the appropriate type*, you can obtain the solution by comparing terms. Suppose the initial conditions and boundary conditions are

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 3\sin(\pi x) - \sin\left(\frac{3\pi x}{2}\right)$$

Set

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{2}\right) = 3\sin(\pi x) - \sin\left(\frac{3\pi x}{2}\right).$$

First, $b_n \sin\left(\frac{\pi n x}{2}\right) = 3\sin(\pi x)$ gives $n = 2$ and $b_2 = 3$.

Likewise, $b_n \sin\left(\frac{\pi n x}{2}\right) = -\sin\left(\frac{3\pi x}{2}\right)$ gives $n = 3$ and $b_3 = -1$.

Thus,

$$u(x,t) = 3\exp(-4\pi^2 t)\sin(\pi x) - \exp(-9\pi^2 t)\sin\left(\frac{3\pi x}{2}\right). \ \square$$

Note that you *can't* use the compare-coefficients method if the initial conditions and boundary conditions are

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = \sin x, \qquad 0 < x < 2$$

Reason: The $sinx$ that appears in $u(x,0)$ has period 2π, whereas the sines in the Fourier expansion of $u(x,0)$ will have period $2L = 4$.

To put it another way, the sine terms in the series have the form $\sin\left(\frac{\pi n x}{2}\right)$, whereas the initial condition is simply $\sin x$. There is no *integer* n for which $x = \frac{\pi n x}{2}$.

In this case, you'd need to compute the Fourier sine expansion for $\sin x$:

$$b_n = \frac{2}{2}\int_0^2 \sin x \sin\left(\frac{\pi n x}{2}\right)\,dx = \frac{1}{\pi n - 2}\sin(\pi n - 2) - \frac{1}{\pi n + 2}\sin(\pi n + 2).$$

The solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{1}{\pi n - 2}\sin(\pi n - 2) - \frac{1}{\pi n + 2}\sin(\pi n + 2)\right)\exp(-\pi^2 n^2 t)\sin\left(\frac{\pi n x}{2}\right). \square$$

b. (Homogeneous boundary conditions.) If $u(x,0) = f(x)$ is not a sum of sines, compute the Fourier sine expansion for the initial distribution.

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = \begin{cases} 2 & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$$

$$b_n = \frac{2}{2}\int_0^1 2\cdot\sin\left(\frac{\pi n x}{2}\right)\,dx - \frac{2}{2}\int_1^2 \sin\left(\frac{\pi n x}{2}\right)\,dx =$$
$$\frac{4}{\pi n} - \frac{6}{\pi n}\cos\left(\frac{\pi n}{2}\right) + \frac{2}{\pi n}(-1)^n.$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4}{\pi n} - \frac{6}{\pi n}\cos\left(\frac{\pi n}{2}\right) + \frac{2}{\pi n}(-1)^n\right)\exp(-\pi^2 n^2 t)\sin\left(\frac{\pi n x}{2}\right). \square$$

c. (Nonhomogeneous boundary conditions.) In this situation, the ends are held at constant temperatures, not necessarily zero. Compute the Fourier sine expansion of the initial transient distribution.

$$u(0,t) = 10, \quad u(2,t) = 30, \qquad t > 0$$

$$u(x,0) = 20 + 20x, \qquad 0 < x < 2$$

The steady-state solution is the line joining the boundary points $(0,10)$, $(2,20)$:

$$v(x) = 10x + 10.$$

Hence, the initial transient distribution is

$$w(x,0) = (20 + 20x) - (10x + 10) = 10x + 10.$$

Compute the Fourier sine expansion for $w(x,0)$:

$$b_n = \frac{2}{2}\int_0^2 (10x + 10)\sin\left(\frac{\pi n x}{2}\right)\,dx = \frac{20}{\pi n} - \frac{60}{\pi n}(-1)^n.$$

Then

$$w(x,t) = \sum_{n=1}^{\infty} \left(\frac{20}{\pi n} - \frac{60}{\pi n}(-1)^n\right) \exp(-\pi^2 n^2 t) \sin\left(\frac{\pi n x}{2}\right).$$

The solution is

$$u(x,t) = w(x,t) = 10x + 10. \square$$

d. (Insulated ends.) In this case, compute the Fourier cosine expansion of the initial distribution.

$$u_x(0,t) = u_x(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 3x + 1, \qquad 0 < x < 2$$

First,

$$a_0 = \frac{2}{2}\int_0^2 (3x+1)\,dx = \left[\frac{3}{2}x^2 + x\right]_0^2 = 8.$$

Next,

$$a_n = \frac{2}{2}\int_0^2 (3x+1)\cos\left(\frac{\pi n x}{2}\right)\,dx = \left[\frac{2}{\pi n}(3x+1)\sin\left(\frac{\pi n x}{2}\right) + \frac{12}{\pi^2 n^2}\cos\left(\frac{\pi n x}{2}\right)\right]_0^2 = -\frac{12}{\pi^2 n^2}\left(1 - (-1)^n\right).$$

The solution is

$$u(x,t) = 4 - \sum_{n=1}^{\infty} \frac{12}{\pi^2 n^2}\left(1 - (-1)^n\right) \exp(-\pi^2 n^2 t) \cos\left(\frac{\pi n x}{2}\right). \square$$

As you can imagine, there are other possibilities — for example, one end is held at a constant temperature while the other is insulated. Or perhaps heat is allowed to escape from one end at a variable rate. The important thing to remember is that all of these cases are derived using **separation of variables**.