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Partial differential equations. Separation of variables.

If u is a function of variables x, y, ..., a **partial differential equation** is an equation of the form

$$F(x, y, \ldots, u; u_x, u_y, \ldots, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0.$$

As with ordinary differential equations, the idea is to find a **general solution**: roughly, a solution containing arbitrary constants, from which every particular solution may be obtained (by assigning values to the constants). However, in the solution of partial differential equations, the "arbitrary constants" are often arbitrary *functions*.

In some cases, solution is straightforward.

Examples.

1. $u_x = 0$, for a function $u(x, y)$.

Since $u_x = 0$, u is constant with respect to x. That is, $u = f(y)$, for some function f. □

2. $u_{xy} = 0$, for a function $u(x, y)$.

Integrate with respect to y to obtain $u_x = \phi(x)$ for some function ϕ. Now integrate with respect to x, keeping in mind that y is now constant:

$$u = \int \phi(x)\, dx + g(y) = f(x) + g(y).$$

For example, $u = \left(x^2 + 1\right)^{117} + \sin e^y$ works. □

In general, it is much harder to solve PDE's than ODE's; it's even difficult to know when you've obtained all the solutions. However, since many things in the real world depend on more than one variable, PDE's are very important in applications. Here, for example, are some equations that one encounters in physics and engineering.

Let

$$\triangle u = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

denote the **Laplacian** of u.

$$\begin{array}{rl} \triangle u = 0 & \text{the Laplace equation} \\ u_{tt} = c^2 \triangle u & \text{the wave equation} \\ \left.\begin{array}{r} \epsilon E_t = \operatorname{curl} H \\ \mu H_t = -\operatorname{curl} E \\ \operatorname{div} E = \operatorname{div} H = 0 \end{array}\right\} & \text{Maxwell's equations} \\ u_t = k \triangle u & \text{the heat equation} \end{array}$$

The method of **separation of variables** is useful in solving some PDE's. The idea is to assume a solution of the form $u(x, y) = f(x) + g(y)$ or $u(x, y) = f(x)g(y)$, then use the given data to determine f and g.

Examples.

1. $u_x^2 + u_y^2 = 25$

Try a solution $u(x, y) = f(x) + g(y)$. $u_x = f'(x)$ and $u_y = g'(y)$, so $f'(x)^2 + g'(y)^2 = 25$, or

$$f'(x)^2 = 25 - g'(y)^2.$$

Here is the idea that characterizes the separation of variables method. The left side depends only on x, while the right side depends only on y. Hence, both sides must be equal to a numerical constant c^2 (positive, since the left side is a square):

$$f'(x)^2 = c^2 = 25 - g'(y)^2.$$

I now have two equations, which I solve separately:

$$f'(x)^2 = c^2, \quad f'(x) = \pm c, \quad f(x) = \pm cx + a.$$

And

$$c^2 = 25 - g'(y)^2, \quad g'(y) = \pm\sqrt{25 - c^2}, \quad g(y) = \pm y\sqrt{25 - c^2} + b.$$

Therefore,

$$u(x, y) = \pm cx \pm y\sqrt{25 - c^2} + a + b = \pm cx \pm y\sqrt{25 - c^2} + d.$$

(I set $d = a + b$, a single arbitrary constant.) All combinations of signs are allowed. For example, $y = 3x - 4y$, $y = -3x + 4y$, $y = 3x + 4y$, and $y = -3x - 4y$ are all solutions of this equation. □

2. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, $u(0, y) = 8e^{-3y}$.

Guess a solution of the form $u = f(x)g(y)$. Then

$$\frac{\partial u}{\partial x} = f'(x)g(y) \quad \text{and} \quad \frac{\partial u}{\partial y} = 4f(x)g'(y) \quad \text{so} \quad f'(x)g(y) = 4f(x)g'(y).$$

Therefore,

$$\frac{f'(x)}{f(x)} = 4\frac{g'(y)}{g(y)}.$$

The left side depends on x, the right side on y. Hence, both sides are constant:

$$\frac{f'(x)}{f(x)} = 4\frac{g'(y)}{g(y)} = c.$$

First,

$$\frac{f'(x)}{f(x)} = c, \quad \ln|f(x)| = cx + c_1, \quad f(x) = d_1 e^{cx}.$$

In addition,

$$4\frac{g'(y)}{g(y)} = c, \quad 4\ln|g(y)| = cy + c_2, \quad g(y) = d_2 \exp\frac{cy}{4}.$$

Therefore,

$$u(x, y) = d_1 d_2 e^{cx} \exp\frac{cy}{4} = d \exp c\left(x + \frac{1}{4}y\right).$$

Plug in the initial condition:

$$8e^{-3y} = u(0, y) = d \exp\frac{cy}{4}.$$

This is true for all y, so set $y = 0$. Then $8 = d$, and

$$8e^{-3y} = 8\exp\frac{cy}{4}, \quad -3y = \frac{1}{4}cy, \quad c = -12.$$

So

$$u(x, y) = 8\exp -12\left(x + \frac{1}{4}y\right) = 8\exp(-12x - 3y). \ \square$$

3. (**One dimensional heat conduction**)

$$u_t = 2u_{xx}, \qquad 0 \le x \le 3$$
$$u(x,0) = 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x), \qquad 0 < x < 3$$
$$0 = u(0,t) = u(3,t), \qquad t > 0$$
$$|u(x,t)| \le M \text{ for all } t$$

This describes heat conduction in a rod of length 3, where the ends of the rod are held at $0°$ and the initial temperature distribution is $5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$. $u(x,t)$ is the temperature at position x at time t. The last condition means that u is bounded — the temperature at a point in the rod should not grow arbitrarily large. (I will often get lazy and omit the statement of this boundedness condition.)

Try $u = f(x)g(t)$. Then

$$\frac{\partial u}{\partial t} = f(x)g'(t), \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = f''(x)g(t) \quad \text{so} \quad f(x)g'(t) = 2f''(x)g(t).$$

Therefore,

$$\frac{g'(t)}{g(t)} = 2\frac{f''(x)}{f(x)}.$$

The left side depends only on t, the right side only on x. Hence, both sides are constant:

$$\frac{g'(t)}{g(t)} = 2\frac{f''(x)}{f(x)} = -\lambda^2.$$

(I'll show later that the boundedness condition implies that the constant is negative.)

First,

$$\frac{g'(t)}{g(t)} = -\lambda^2, \quad g(t) = c_0 \exp\left(-\lambda^2 t\right).$$

(Notice that if I'd used a positive constant λ^2 instead, the exponential would go to ∞ as $t \to \infty$, contradicting the boundedness condition.)

Next,

$$2\frac{f''(x)}{f(x)} = -\lambda^2 \quad \text{so} \quad f''(x) + \frac{\lambda^2}{2}f(x) = 0.$$

Therefore,

$$f(x) = c_1 \cos\frac{\lambda}{\sqrt{2}}x + c_2 \sin\frac{\lambda}{\sqrt{2}}x.$$

Thus,

$$u(x,t) = c_0 \exp\left(-\lambda^2 t\right)\left(c_1 \cos\frac{\lambda}{\sqrt{2}}x + c_2 \sin\frac{\lambda}{\sqrt{2}}x\right) = \exp\left(-\lambda^2 t\right)\left(d_1 \cos\frac{\lambda}{\sqrt{2}}x + d_2 \sin\frac{\lambda}{\sqrt{2}}x\right).$$

Insert the boundary conditions. $0 = u(0,t) = \exp\left(-\lambda^2 t\right) \cdot d_1$ gives $d_1 = 0$. Therefore,

$$u(x,t) = d_2 \exp\left(-\lambda^2 t\right) \sin\frac{\lambda}{\sqrt{2}}x.$$

$$0 = u(3,t) = d_2 \exp\left(-\lambda^2 t\right) \sin\frac{3\lambda}{\sqrt{2}}, \quad 0 = d_2 \sin\frac{3\lambda}{\sqrt{2}}.$$

If $d_2 = 0$, I get the trivial solution $u = 0$. If $d_2 \ne 0$,

$$0 = \sin\frac{3\lambda}{\sqrt{2}}, \quad \lambda = \frac{n\pi\sqrt{2}}{3}.$$

Therefore,

$$u(x,t) = d_2 \exp\left(-\frac{2n^2\pi^2 t}{9}\right) \sin\left(\frac{n\pi}{3} x\right).$$

However, there's a problem when I plug in the initial condition:

$$5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x) = u(x,0) = d_2 \sin\left(\frac{n\pi}{3} x\right).$$

There is no constant d_2 which satisfies this equation.

Instead, I'll use the **principle of superposition**: If u_1 and u_2 solve $u_t = 2u_{xx}$ and satisfy the boundary conditions, so does their sum $u_1 + u_2$. This is true because the original equation is *linear*.

Therefore, I'll break up the initial condition into the following three equations, solve them separately, then add the solutions:

$$5\sin(4\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

$$-3\sin(8\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

$$2\sin(10\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

First,

$$5\sin(4\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

gives $d_2 = 5$ $n = 12$. Therefore,

$$u_1 = 5\exp\left(-32\pi^2 t\right) \sin\left(4\pi x\right).$$

Next,

$$-3\sin(8\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

gives $d_2 = -3$ and $n = 24$. Therefore,

$$u_2 = -3\exp\left(-128\pi^2 t\right) \sin\left(8\pi x\right).$$

Finally,

$$2\sin(10\pi x) = d_2 \sin\left(\frac{n\pi}{3} x\right)$$

gives $d_2 = 2$ and $n = 30$. Therefore,

$$u_3 = 2\exp\left(-200\pi^2 t\right) \sin\left(10\pi x\right).$$

The solution is

$$u(x,t) = 5\exp\left(-32\pi^2 t\right) \sin\left(4\pi x\right) - 3\exp\left(-128\pi^2 t\right) \sin\left(8\pi x\right) + 2\exp\left(-200\pi^2 t\right) \sin\left(10\pi x\right). \square$$

Rather than go through the superposition argument every time, I'll summarize the general situation as follows. Suppose you are given a heat conduction problem where the ends of the medium are held at 0° and the initial temperature distribution is $f(x)$:

$$u_t = a^2 u_{xx}, \qquad 0 \le x \le L$$

$$u(0,t) = u(L,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \qquad 0 < x < L$$

Then the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{a^2 n^2 \pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right).$$

In the example, only finitely many of the terms were nonzero, those terms — three of them — being determined by the initial temperature distribution.

Example. Solve the heat conduction problem

$$u_t = 25 u_{xx}, \qquad 0 \le x \le 2$$

$$u(0,t) = u(2,t) = 0, \qquad t > 0$$

$$u(x,0) = 3\sin(7\pi x), \qquad 0 < x < 2$$

$0 \le x \le 2$ tells us that $L = 2$; the initial equation implies that $a^2 = 25$. Then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{25 n^2 \pi^2 t}{4}\right) \sin\left(\frac{n\pi x}{2}\right).$$

Let $t = 0$:

$$3\sin 7\pi x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right).$$

I'll get $\sin 7\pi x$ when $7\pi x = \dfrac{n\pi x}{2}$, or $n = 14$. Therefore, $b_{14} = 3$. There are no other sine terms in $u(x,0)$, so $b_n = 0$ for $n \ne 14$. The solution is

$$u(x,t) = 3\exp(-1225\pi^2 t)(\sin 7\pi x). \quad \square$$

Note that a sum of finitely many periodic functions will be periodic, so it seems as though a finite sum will only account for periodic initial conditions. An arbitrary initial temperature distribution would seem to require infinitely many nonzero terms in the sum. This leads to series of the form $\sum b_n \sin(n\pi x)$ having infinitely many nonzero terms — **Fourier series**.