Math 227
10-20-1993

Review Session - Test 2.

These problems appeared on previous Math 227 exams.

1. Find the area of the loop

$$x = \frac{t^{1/2}}{(1+t)^3}, \quad y = \frac{t^{5/2}}{(1+t)^3}, \qquad 0 \le t < +\infty.$$

By Green's theorem, the area of the loop is

$$A = \int_{\vec{\sigma}} x\, dy.$$

Compute $\frac{dy}{dt}$ and simplify it:

$$\frac{dy}{dt} = \frac{(1+t)^3 \left(\frac{5}{2}\right) t^{3/2} - t^{5/2}(3)(1+t)^2}{(1+t)^6} = \frac{(1+t)\left(\frac{5}{2}\right) t^{3/2} - 3t^{5/2}}{(1+t)^6} = \frac{1}{2} t^{3/2} \frac{5-t}{(1+t)^4}.$$

Hence, the area is

$$A = \int_0^{+\infty} \frac{1}{2} \frac{t^2(5-t)}{(1+t)^7}\, dt.$$

Let $u = 1+t$, so $du = dt$. When $t = 0$, $u = 1$; when $t = \infty$, $u = \infty$. Therefore,

$$A = \int_1^{\infty} \frac{1}{2} \frac{(u-1)^2(6-u)}{u^7}\, du = \frac{1}{2} \int_1^{\infty} \left(\frac{6}{u^7} - \frac{13}{u^6} + \frac{8}{u^5} - \frac{1}{u^4}\right) du =$$

$$\frac{1}{2} \lim_{c \to \infty} \left[-\frac{1}{u^6} + \frac{13}{5u^5} - \frac{2}{u^4} + \frac{1}{3u^3}\right]_1^c = \frac{1}{2}\left[1 - \frac{13}{5} + 2 - \frac{1}{3}\right] = \frac{1}{30}. \square$$

2. Determine whether the field

$$\vec{F} = \langle 6x^2 e^{yz}, 2x^3 z e^{yz} - 3y^2, 2x^3 y e^{yz} \rangle$$

is conservative. If it is, find a function f such that $\nabla f = \vec{F}$. If it isn't, *prove* that it isn't.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2 e^{yz} & 2x^3 z e^{yz} - 3y^2 & 2x^3 y e^{yz} \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

Therefore, the field is conservative. A potential function f must satisfy

$$\frac{\partial f}{\partial x} = 6x^2 e^{yz}, \quad \frac{\partial f}{\partial y} = 2x^3 z e^{yz} - 3y^2, \quad \frac{\partial f}{\partial z} = 2x^3 y e^{yz}.$$

Integrate $\frac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x} = 6x^2 e^{yz}, \text{ so } f = 2x^3 e^{yz} + C(y, z).$$

Differentiate with respect to y and set the result equal to $\dfrac{\partial f}{\partial y}$:

$$2x^3 z e^{yz} - 3y^2 = \frac{\partial f}{\partial y} = 2x^3 z e^{yz} + \frac{\partial C}{\partial y}.$$

Hence, $\dfrac{\partial C}{\partial y} = -3y^2$ and $C = -y^3 + D(z)$. So

$$f = 2x^3 e^{yz} - y^3 + D(z).$$

Differentiate with respect to z and set the result equal to $\dfrac{\partial f}{\partial z}$:

$$2x^3 y e^{yz} = \frac{\partial f}{\partial z} = 2x^3 y e^{yz} + \frac{dD}{dz}.$$

$\dfrac{dD}{dz} = 0$, so D is a numerical constant. I may take $D = 0$. Hence,

$$f = 2x^3 e^{yz} - y^3.\ \square$$

3. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$, where $\vec{F} = \langle y^2, 2xy + 2yz^3, 3y^2 z^2 \rangle$ and $\vec{\sigma}(t) = \left\langle \sqrt{5t^3 + 4}, \dfrac{2}{t+1}, t^2 + 1 \right\rangle$, $0 \le t \le 1$.

The fact that the path is ugly suggests looking for an easier approach than a direct computation. In fact, the field is a gradient field: If $f = xy^2 + y^2 z^3$, then $\nabla f = \vec{F}$. All I have to do is evaluate f at the endpoint of the path.

$\vec{\sigma}(0) = (2, 2, 1)$ and $\vec{\sigma}(1) = (3, 1, 2)$, so

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = f(3, 1, 2) - f(2, 2, 1) = 11 - 12 = -1.\ \square$$

4. Compute the circulation of $\vec{F} = \langle x^2 y, xz, y^2 z \rangle$ around the boundary of the rectangle with vertices $(0, 0, 1)$, $(2, 0, 1)$, $(0, 2, 3)$, $(2, 2, 3)$, where the boundary is traversed in the counterclockwise direction as viewed from above.

By Stokes' theorem, the circulation around the boundary of the rectangle equals the flux of curl $\vec{F}$ through the surface. First,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & xz & y^2 z \end{vmatrix} = \langle -x + 2yz, 0, -x^2 + z \rangle.$$

You can parametrize the surface as a parallelogram, but it's easier to notice that $z = y + 1$ for all four points — so in fact, the rectangle is part of the plane $z = y + 1$. The projection into the x-y-plane is

$$0 \le x \le 2, \quad 0 \le y \le 2.$$

The normal vector is

$$\vec{N} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle 0, -1, 1 \rangle.$$

Note that this is the upward normal, consistent with the orientation of the boundary.

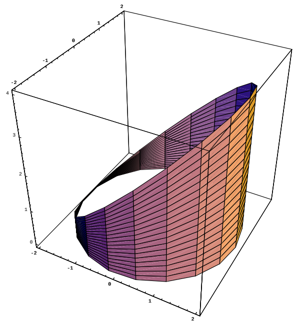
Hence,

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -x^2 + z = -x^2 + y + 1.$$

Therefore, if R is the rectangle, the circulation is

$$\int_{\partial R} \vec{F} \cdot \overrightarrow{ds} = \iint_R \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_0^2 \int_0^2 -x^2 + y + 1 \, dx \, dy = \frac{8}{3}. \square$$

5. Find the flux of $\vec{F} = \langle xz, yz, 3 \rangle$ through the part of the cylinder $x^2 + y^2 = 4$ which lies between $z = 0$ and $z = x + 2$, using the cylinder's *outward* normal.



The problem asks for the flux of $\vec{F}$, *not* the flux of $\operatorname{curl} \vec{F}$, so you *cannot* use Stokes' theorem!

The hard part is to parametrize the surface, which is part of the cylinder $x^2 + y^2 = 4$. To do this, decompose the surface into vertical segments. The bottom end of a segment is a point on $x^2 + y^2 = 4$, which is $(2\cos u, 2\sin u, 0)$. The top end of a segment has the same x and y coordinates as the bottom, but $z = x + 2$; it is $(2\cos u, 2\sin u, 2\cos u + 2)$.

The segment connecting these points is

$$(x, y, z) = (1 - v) \cdot (2\cos u, 2\sin u, 0) + v \cdot (2\cos u, 2\sin u, 2\cos u + 2) = (2\cos u, 2\sin u, v(2\cos u + 2)).$$

The parameter ranges are

$$0 \le u \le 2\pi, \quad 0 \le v \le 1.$$

The normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin u & 2\cos u & -2v\sin u \\ 0 & 0 & 2\cos u + 2 \end{vmatrix} = \langle 2\cos u(2\cos u + 2), 2\sin u(2\cos u + 2), 0 \rangle.$$

Convert $\vec{F}$:

$$\vec{F} = \langle 2v\cos u(2\cos u + 2), 2v\sin u(2\cos u + 2), 3 \rangle.$$

Hence,

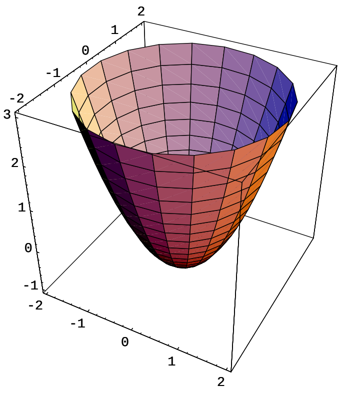
$$\vec{F} \cdot \vec{N} = 4v(\cos u)^2(2\cos u + 2)^2 + 4v(\sin u)^2(2\cos u + 2)^2 = 4v(2\cos u + 2)^2.$$

The flux is

$$\int_0^{2\pi} \int_0^1 4v(2\cos u + 2)^2 \, dv \, du = \int_0^{2\pi} 2(2\cos u + 2)^2 \, du =$$

$$8 \int_0^{2\pi} \left(1 + 2\cos u + \frac{1}{2}(1 + \cos 2u)\right) du = 24\pi \approx 75.39822. \square$$

6. Let $\vec{F} = \langle z^2 - y, 2xy, 2x^4y^2 \rangle$. Find the flux of curl $\vec{F}$ through the surface $z = x^2 + y^2 - 1$, $z \le 0$, using the normal with positive z-component.



By Stokes' theorem, the flux of curl $\vec{F}$ is the line integral of $\vec{F}$ around the boundary of the surface. The boundary is the unit circle $x^2 + y^2 = 1$, and computing the circulation is not that difficult.

However, by Stokes' theorem I may replace the surface with any other surface having the same boundary. The unit disk $x^2 + y^2 \le 1$, $z = 0$, also has $x^2 + y^2 = 1$ as its boundary. The normal for $z = 0$ is $\vec{N} = \langle 0, 0, 1 \rangle$. (This is the upward normal, since the z component is positive.)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 - y & 2xy & 2x^4y^2 \end{vmatrix} = \langle 4x^4y, -8x^3y^2 + 2z, 1 + 2y \rangle.$$

Therefore, curl $\vec{F} \cdot \vec{N} = 1 + 2y$.

Convert to polar. $1 + 2y = 1 + 2r\sin\theta$, and the region of integration is the unit disk $x^2 + y^2 \le 1$:

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$$

Therefore, the flux of curl $\vec{F}$ is

$$\int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) r\,dr\,d\theta = \pi. \ \Box$$