Math 227
11-20-1993

Review Session for Test 3.

1.

The constants are $L = 2$ and $\alpha^2 = 16$.

$$a_0 = \frac{2}{2}\int_0^2 |x-1|\,dx = 1.$$

Next,

$$a_n = \frac{2}{2}\int_0^2 u(x,0)\cos\frac{\pi n x}{2}\,dx = \int_0^1 (1-x)\cos\frac{\pi n x}{2}\,dx + \int_1^2 (x-1)\cos\frac{\pi n x}{2}\,dx.$$

The same integration by parts can be used for both integrals.

$$\begin{array}{cccc}
 & \dfrac{d}{dx} & & \int dx \\
+ & x-1 & & \cos\dfrac{\pi n x}{2} \\
 & & \searrow & \\
- & 1 & & \dfrac{2}{\pi n}\sin\dfrac{\pi n x}{2} \\
 & & \searrow & \\
+ & 0 & & -\dfrac{4}{\pi^2 n^2}\cos\dfrac{\pi n x}{2}
\end{array}$$

So a_n is

$$\left[\frac{2}{\pi n}(1-x)\sin\frac{\pi n x}{2} - \frac{4}{\pi^2 n^2}\cos\frac{\pi n x}{2}\right]_0^1 + \left[\frac{2}{\pi n}(x-1)\sin\frac{\pi n x}{2} + \frac{4}{\pi^2 n^2}\cos\frac{\pi n x}{2}\right]_1^2 = \frac{4}{\pi^2 n^2}\left(1 - \cos\frac{\pi n}{2}\right).$$

The solution is

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left(1 - \cos\frac{\pi n}{2}\right)\exp(-4\pi^2 n^2 t)\cos\frac{\pi n x}{2}.$$

Note that as $t \to \infty$, $u \to \dfrac{1}{2}$. $\square$

3. Solve the wave equation

$$\begin{gathered}
u_{tt} = 4u_{xx}, \qquad 0 < x < 2, \quad t > 0 \\
u(0,t) = u(2,t) = 0, \qquad t > 0 \\
u(x,0) = \begin{cases} -x & \text{if } 0 \le x < 1 \\ x-2 & \text{if } 1 \le x \le 2 \end{cases} \\
u_t(x,0) = 1, \qquad 0 < x < 2
\end{gathered}$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{\pi n x}{L}\left(b_n \cos\frac{\pi n a t}{L} + k_n \sin\frac{\pi n a t}{L}\right).$$

b_n is the n-th Fourier sine coefficient of $u(x,0)$ and $\dfrac{\pi n a}{L}k_n$ is the n-th Fourier sine coefficient of $u_t(x,0)$. In our problem, $a = 2$ and $L = 2$.

First,

$$b_n = \frac{2}{2}\left(\int_0^1 (-x)\sin\frac{\pi n x}{2}\,dx + \int_1^2 (x-2)\sin\frac{\pi n x}{2}\,dx\right) =$$

$$\left[\frac{2}{\pi n}x\cos\frac{\pi n x}{2} - \frac{4}{\pi^2 n^2}\sin\frac{\pi n x}{2}\right]_0^1 + \left[-\frac{2}{\pi n}(x-2)\cos\frac{\pi n x}{2} + \frac{4}{\pi^2 n^2}\sin\frac{\pi n x}{2}\right]_1^2 = -\frac{8}{\pi^2 n^2}\sin\frac{\pi n}{2}.$$

Next,

$$\frac{\pi n \cdot 2}{2} k_n = \frac{2}{2} \int_0^2 (1) \sin \frac{\pi n x}{2}\, dx = \left[-\frac{2}{\alpha\alpha\alpha\alpha n} \cos \frac{\pi n x}{2} \right]_0^2 = \frac{2}{\pi n}(1 - \cos \pi n).$$

Therefore,

$$k_n = \frac{2}{\pi^2 n^2}(1 - \cos \pi n).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{\pi n x}{2} \left(-\frac{8}{\pi^2 n^2} \sin \frac{\pi n}{2} \cos(\pi n t) + \frac{2}{\pi^2 n^2}(1 - \cos \pi n) \sin(\pi n t) \right). \; \square$$

4. Use separation of variables to find a formal solution to the wave equation

$$u_{tt} = a^2 u_{xx}, \qquad 0 < x < L, \quad t > 0$$

$$u(0,t) = 0, \quad u_x(L,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \qquad 0 < x < L$$

Hence,

$$0 = u_x(L, t) = -\frac{\lambda}{a} \cdot d_2 \cos \frac{\lambda L}{a}.$$

Then $\dfrac{\lambda L}{a}$ must be an odd multiple of π:

$$\frac{\lambda L}{a} = (2n - 1)\pi, \quad \lambda = \frac{\pi(2n - 1)a}{L}, \qquad n = 1, 2, 3, \ldots.$$

So

$$p(x) = d_2 \sin \frac{\pi(2n - 1)x}{L} \text{ while } q(t) = c_1$$

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ff :

You can get the general solution by superposition (summing from $n = 1$ to ∞.)

It is a separate issue (and one I won't consider) to determine how the coefficients in the series are computed. Briefly put, one must extend the initial displacement function outside the interval 0 to L in a certain way, then expand the extended function — like the odd extension for a Fourier sine series, but a little different. □