9-20-1993

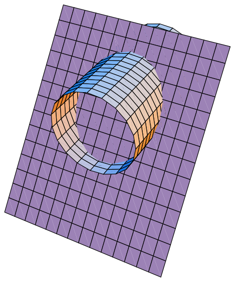
Review - Curves.

Example. Parametrize the curve of intersection of

$$y^2 + z^2 = 4 \quad \text{and} \quad x - 2y + z = 3$$

without using radicals. What is the curve?

A circular cylinder ($y^2 + z^2 = 4$) should intersect a plane ($x - 2y + z = 3$) in an ellipse.



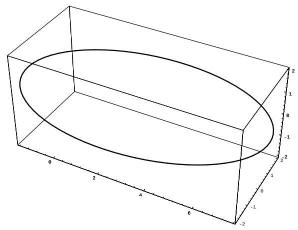
If I project the curve along the cylinder into the y-z plane, I get the curve $y^2 + z^2 = 4$. It can be parametrized by

$$y = 2\cos t, \quad z = 2\sin t, \qquad 0 \le t \le 2\pi.$$

Solve the plane equation for x: $x = 2y - z + 3$. Substitute for y and z: $x = 4\cos t - 2\sin t + 3$.

The curve of intersection is

$$x = 4\cos t - 2\sin t + 3, \quad y = 2\cos t, \quad z = 2\sin t, \qquad 0 \le t \le 2\pi.$$



$\square$

When is a curve actually a line? It turns out that this is true if $\vec{\sigma}\,'(t) \times \vec{\sigma}\,''(t) = \vec{0}$ but $\vec{\sigma}\,'(t) \ne \vec{0}$ on an interval $a < t < b$. Why is this true?

You can prove this formally by using results on differential equations. However, it's easy to see why this is plausible if you imagine the curve as being traced out by a moving particle.

Recall that nonzero vectors $\vec{a}$, $\vec{b}$ are collinear if $\vec{a} \times \vec{b} = \vec{0}$. Hence, the condition that $\vec{\sigma}\,'(t) \times \vec{\sigma}\,''(t) = \vec{0}$ means that the velocity $\vec{\sigma}\,'(t)$ and the acceleration $\vec{\sigma}\,''(t)$ lie along the same line.

For a small time increment dt, the velocity goes from $\vec{\sigma}\,'(t)$ to $\vec{\sigma}\,'(t) + \vec{\sigma}\,''(t) \cdot dt$, because the acceleration is the rate of change of the velocity. But if $\vec{\sigma}\,'(t)$ and $\vec{\sigma}\,''(t)$ lie along the same line, the new velocity $\vec{\sigma}\,'(t) + \vec{\sigma}\,''(t) \cdot dt$ lies along the same line. Hence, the particle should continue moving along the line.

Example. Consider the curve

$$\vec{\sigma}(t) = \langle 1 + \cos t, 2\cos t, 1 - 2\cos t\rangle.$$

The velocity and acceleration are

$$\vec{\sigma}\,'(t) = \langle -\sin t, -2\sin t, 2\sin t\rangle, \quad \vec{\sigma}\,''(t) = \langle -\cos t, -2\cos t, 2\cos t\rangle.$$

Hence,

$$\vec{\sigma}\,'(t) \times \vec{\sigma}\,''(t) = \left|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin t & -2\sin t & 2\sin t \\ -\cos t & -2\cos t & 2\cos t \end{matrix}\right| = \langle 0, 0, 0\rangle.$$

Therefore, the curve is linear *at least* on any interval where the velocity isn't $\vec{0}$ — for example, on $0 < t < \pi$.

Of course, this is evident from the original equation: If you set $u = \cos t$, the equation becomes

$$\vec{\sigma}(u) = \langle 1 + u, 2u, 1 - 2u\rangle.$$

This has the form of a parametric equation of a line. Note that since $u = \cos t$ only ranges from -1 to 1, the curve is not an entire line — just a segment. □

Example. Calvin Butterball's space ship is at $\vec{\sigma}(t) = \left\langle 2e^{t-1}, \dfrac{1}{t^2}, t^3\right\rangle$ at times t. At $t = 1$, the hamster dies, the engine stops, and the ship flies off along the tangent line to the curve. Ignoring gravity, at what point does the ship hit the x-y plane, smashing Calvin to a blody pulp?

First, I'll find the equation of the tangent line.

$$\vec{\sigma}\,'(t) = \left\langle 2e^{t-1}, -\frac{2}{t^3}, 3t^2\right\rangle, \quad \vec{\sigma}\,'(1) = \langle 2, -2, 3\rangle.$$

The point of tangency is $\vec{\sigma} = \langle 2, 1, 1\rangle$. The tangent line is

$$x - 2 = 2t, \quad y - 1 = -2t, \quad z - 1 = 3t.$$

Next, I have to find where the line intersects the x-y plane ($z = 0$). Set $z = 0$; the last equation gives $t = -\dfrac{1}{3}$. If I plug this back into the x and y equations, I get $x = \dfrac{4}{3}$ and $y = \dfrac{5}{3}$.

The ship hits the x-y plane at the point $\left(\dfrac{4}{3}, \dfrac{5}{3}, 0\right)$. □