11-8-1993

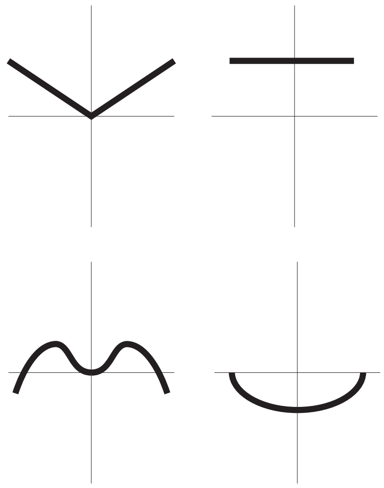
Fourier sine and cosine series.

The title refers to two situations:

- In some cases, the Fourier series of a function will contain no sine terms or no cosine terms.
- You often need to express a function $f(x)$ as an infinite sum of sines or of cosines — as opposed to a Fourier series, which is an infinite sum of sines *and* cosines. (For example, the solution to the heat equation with homogeneous boundary conditions requires the expansion of the initial distribution $u(x, 0)$ as a sum of sines.)

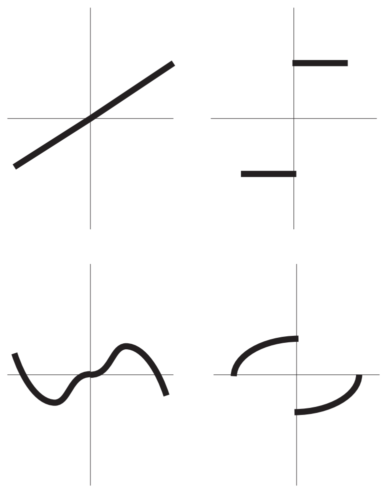
The common idea is that cosines are **even functions** while sines are **odd functions**.

Let me remind you of what **even** and **odd** mean. A function $f(x)$ is **even** if $f(-x) = f(x)$ for all x. This means that the graph of the function is symmetric about the y-axis. Here are some even functions:



$\cos ax$ is an important example of an even function.

A function $f(x)$ is **odd** if $f(-x) = -f(x)$ for all x. This means that the graph of the function is symmetric about the origin. Here are some odd functions:



$\sin ax$ is an important example of an odd function.

Here are some important facts about products of odd and even functions:

1. The product of two odd functions is even.
2. The product of two even functions is even.
3. The product of an odd function and an even function is odd.

(Actually, I will only use the first two facts below.) Let me prove the first fact by way of example. If $f(x)$ and $g(x)$ are odd, here is what happens when you feed $-x$ into their product:

$$(fg)(-x) = f(-x) \cdot g(-x) = [-f(x)] \cdot [-g(x)] = f(x)g(x) = (fg)(x).$$

The second equality used the fact that f and g were odd. Since $(fg)(-x)$ and $(fg)(x)$ are equal, the product function fg is even. You can prove the other two facts in similar fashion.

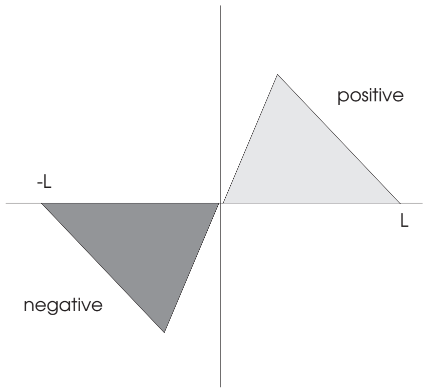
Here is how this applies to Fourier series.

1. The Fourier series of an even function contains only cosine terms; that is, $b_n = 0$ for all n.
2. The Fourier series of an odd function contains only sine terms; that is, $a_n = 0$ for all n.

I'll prove the second result by way of example. Suppose $f(x)$ is odd. Then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{\pi n x}{L} \, dx.$$

But f is odd and $\cos \dfrac{\pi n x}{L}$ is even, so the product is odd. *Now the integral of an odd function over a symmetric ($-L \le x \le L$) interval around the origin is* 0. Reason: Because the function is odd, a positive area on one side is exactly cancelled by a negative area on the other side.



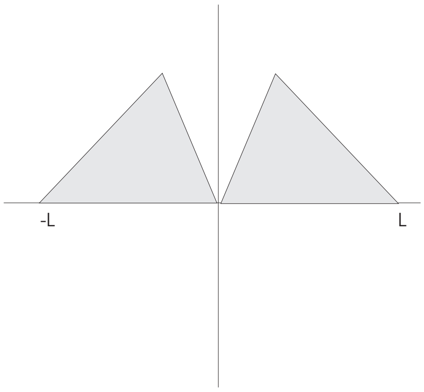
Therefore, $a_n = 0$.

I can use similar reasoning to derive a useful formula for the b_n's in this case.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{\pi n x}{L} \, dx.$$

f is odd and $\sin \dfrac{\pi n x}{L}$ is odd, so their product is even. *But the integral of an even function from* $-L$ *to*

L is two times the integral from 0 to L, by symmetry.



Therefore,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{\pi n x}{L}\,dx.$$

This formula is a *small* win, since plugging in 0 usually makes for easier algebra. But if you're worried about getting the formula right, you can just use the $-L$ to l formula above. It doesn't make *that much* of a difference.

A similar result holds if f is even. I'll summarize everything in a table.

f is even	f is odd
$a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{\pi n x}{L}\,dx$	$a_n = 0$
$b_n = 0$	$b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{\pi n x}{L}\,dx$

Example. Let

$$f(x) = 3x, \qquad -1 \le x < 1, \qquad f(x) = f(x+2) \text{ for all } x.$$

f is odd; you can see this by drawing the graph, for instance. Therefore, the Fourier series has only sine terms. *This means that I only need to compute the b_n's; the a_n's are all 0.* Hence, I'll use the b_n formula rather than the complex exponential.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\pi n x\,dx.$$

By the discussion above, I could also use

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin\pi n x\,dx.$$

I'll use the second form:

$$b_n = 2\int_0^1 (3x)\sin\pi n x\,dx = 6\left[-\frac{1}{\pi n}\cos\pi n x + \frac{1}{\pi^2 n^2}\sin\pi n x\right]_0^1 = -\frac{6}{\pi n}\cos\pi n.$$

The Fourier series is

$$f(x) = \sum_{n=1}^{\infty} -\frac{6}{\pi n}\cos\pi n\sin\pi n x. \ \square$$

Now let $f(x)$ be defined on the interval $0 \le x < L$. Suppose I want a Fourier expansion of $f(x)$ involving only sine terms. I construct the **odd extension** of f by defining

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x < L \\ -f(-x) & \text{if } -L \le x < 0 \end{cases}, \qquad g(x) = g(x + 2L) \text{ for all } x.$$

The first part of the definition reflects f about the origin. This gives an odd function on $-L$ to L. The condition $f(x) = f(x + 2L)$ just repeats the definition to the left and right to get a periodic function.

I get the **Fourier sine series** for f by computing the Fourier expansion of the odd extension g. Of course, it will have only sine terms. The discussion above shows that the sine coefficients are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{\pi n x}{L}\, dx.$$

Notice that this makes no reference to the odd extension; it only requires the given definition of f on the interval 0 to L.

Likewise, if $f(x)$ is defined on the interval $0 \le x < L$ and I want a Fourier expansion of $f(x)$ involving only cosine terms, I construct the **even extension** of f by defining

$$h(x) = \begin{cases} f(x) & \text{if } 0 \le x < L \\ f(-x) & \text{if } -L \le x < 0 \end{cases}, \qquad h(x) = h(x + 2L) \text{ for all } x.$$

The **Fourier cosine series** for f is the Fourier expansion of the even extension h. Of course, it will have only cosine terms. The discussion above shows that the cosine coefficients are given by

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{\pi n x}{L}\, dx.$$

Notice that this makes no reference to the even extension; it only requires the given definition of f on the interval 0 to L.

Example. Consider the function $f(x) = 3 - x$ defined on the interval $0 \le x < 3$. The Fourier cosine coefficients for f are the Fourier coefficients for

$$g(x) = \begin{cases} 3 - x & \text{if } 0 \le x < 3 \\ 3 + x & \text{if } -3 \le x < 0 \end{cases}, \qquad g(x) = g(x + 6) \text{ for all } x.$$

The Fourier sine coefficients for f are the Fourier coefficients for

$$h(x) = \begin{cases} 3 - x & \text{if } 0 \le x < 3 \\ -3 - x & \text{if } -3 \le x < 0 \end{cases}, \qquad h(x) = h(x + 6) \text{ for all } x.$$

This really does define h for all x! For example,

$$h(17) = h(11) = h(5) = h(-1) = -3 - 1 = -4.$$

Since h is periodic with period 6, I just march backwards by steps of 6 units until I wind up in the interval $-3 \le x < 3$, where I can apply the formulas. □

Examples. 1. Find the Fourier cosine series for

$$f(x) = -2 - x, \qquad -2 \le x < 0.$$

First, compute the a_0 term:

$$a_0 = \frac{2}{2} \int_{-2}^0 f(x)\, dx = \int_{-2}^0 (-2 - x)\, dx = \left[-2x - \frac{1}{2}x^2\right]_{-2}^0 = -2.$$

Next,

$$a_n = \frac{2}{2}\int_{-2}^{0} f(x)\cos\left(\frac{\pi n x}{2}\right)\,dx = \int_{-2}^{0}(-2-x)\cos\left(\frac{\pi n x}{2}\right)\,dx =$$

$$\left[\frac{2}{\pi n}(-2-x)\sin\left(\frac{\pi n x}{2}\right)\frac{4}{\pi^2 n^2}\cos\left(\frac{\pi n x}{2}\right)\right]_{-2}^{0} = -\frac{4}{\pi^2 n^2}\left(1-(-1)^n\right).$$

The Fourier cosine series for f is

$$f(x) = -1 - \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left(1-(-1)^n\right)\cos\left(\frac{\pi n x}{2}\right). \quad \square$$

2. Find the Fourier sine series for

$$f(x) = x^2, \qquad 0 < x < 1.$$

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin(\pi n x)\,dx = 2\int_0^1 x^2\sin(\pi n x)\,dx =$$

$$2\left[-\frac{1}{\pi n}x^2\cos(\pi n x) + \frac{2}{\pi^2 n^2}x\sin(\pi n x) + \frac{2}{\pi^3 n^3}\cos(\pi n x)\right]_0^1 = \frac{2}{\pi n}(-1)^{n+1} - \frac{2}{\pi^3 n^3}\left(1-(-1)^n\right).$$

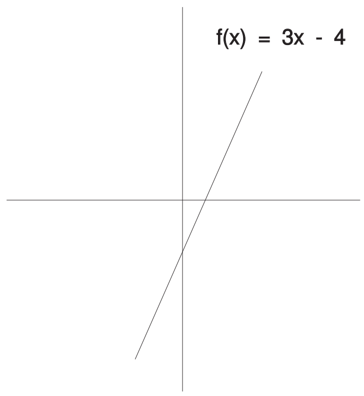
The Fourier sine series for $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}(-1)^{n+1} - \frac{2}{\pi^3 n^3}\left(1-(-1)^n\right)\right)\sin(\pi n x). \quad \square$$

3. The Fourier expansion of an odd function has no cosine terms, while the Fourier expansion of an even function has no sine terms. In some cases, you can exploit these facts even if the given function is neither even nor odd.

Consider the function

$$f(x) = 3x - 4, \qquad -1 \le x < 1.$$



f is neither even nor odd. However, the function $g(x) = 3x$ is odd, so the Fourier expansion for g has only sine terms. I found in an earlier example that

$$3x = \sum_{n=1}^{\infty} -\frac{6}{\pi n}\cos \pi n \sin \pi n x.$$

Therefore, the expansion for $f(x) = 3x - 4$ is

$$f(x) = -4 + \sum_{n=1}^{\infty} -\frac{6}{\pi n} \cos \pi n \sin(\pi n x). \square$$

4. The Fourier series for f may be integrated term-by-term provided that f(x) is piecewise continuous in $[-L, L]$. This will be the case with most of the series you encounter.

The new series will converge (uniformly) to the integral

$$\int_a^x f(t)\, dt$$

as long as a and x are both in $[-L, L]$.

As an example, I'll compute the series for $x^2 + x$ on $[-1, 1]$. The computation for $3x$ shows that the series for x is

$$x = \sum_{n=1}^{\infty} -\frac{2}{\pi n} \cos \pi n \sin(\pi n x).$$

Integrating term-by-term,

$$\frac{1}{2} x^2 = \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \cos \pi n \cos(\pi n x) + C.$$

Hence,

$$x^2 = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos \pi n \cos(\pi n x) + C_0.$$

By direct computation, $a_0 = \dfrac{2}{3}$. Therefore, $C_0 = \dfrac{1}{3}$.

Hence,

$$x^2 + x = \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi^2 n^2} \cos \pi n \cos(\pi n x) - \frac{2}{\pi} \cos \pi n \sin(\pi n x) \right). \square$$

5. Solve the heat conduction problem

$$u_t = u_{xx}, \qquad 0 < x < 2, \quad t > 0$$

$$u(0, t) = u(2, t) = 0, \qquad t > 0$$

$$u(x, 0) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x < 2 \end{cases}$$

The solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{\pi^2 n^2 a^2 t}{L^2} \right) \sin\left(\frac{\pi n x}{L} \right).$$

Inserting the initial condition,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{2} \right).$$

This means that I need the Fourier sine series for

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x < 2 \end{cases}$$

So

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{\pi n x}{2}\right)\,dx = \int_0^1 x\sin\left(\frac{\pi n x}{2}\right)\,dx + \int_1^2 (2-x)\sin\left(\frac{\pi n x}{2}\right)\,dx =$$

$$\left[-\frac{2}{\pi n}x\cos\left(\frac{\pi n x}{2}\right) + \frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)\right]_0^1 + \left[-\frac{2}{\pi n}(2-x)\cos\left(\frac{\pi n x}{2}\right) - \frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)\right]_1^2 =$$

$$\frac{8}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right).$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right)\exp\left(-\frac{\pi^2 n^2 t}{4}\right)\sin\left(\frac{\pi n x}{2}\right). \square$$