9-21-1993

The normal vector to a parametrized surface.

If $z = f(x, y)$ is a surface given as the graph of a function, then

$$\vec{N} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

is a normal vector to the surface — a vector perpendicular to the surface at a given point. You can use the normal vector to find, e.g., the equation of the tangent plane to the surface at a point.

Example. Find the equation of the tangent plane to $z = x^2 + y^2$ at the point $(1, 2, 5)$.

$$\vec{N} = \langle -2x, -2y, 1 \rangle, \text{ so } \vec{N}(1, 2, 5) = \langle -1, -4, 1 \rangle.$$

The tangent plane is

$$(-1)(x - 1) + (-4)(y - 2) + (1)(z - 5) = 0, \text{ or } x + 4y - z = 4. \ \square$$

If the surface is given implicitly — i.e. in the form $f(x, y, z) = 0$ — you can find the normal vector by regarding it as a gradient. Recall that the gradient vector ∇w of a function $w = f(x, y, z)$ is perpendicular to the level surface passing through a point. So if the given surface is $f(x, y, z) = 0$, set $w = f(x, y, z)$. Then ∇w will be perpendicular to the level surface $w = 0$, which is the original surface.

Example. Find the equation of the tangent plane to $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$.

Write $w = x^2 + y^2 + z^2 - 3$. Then

$$\nabla w = \langle 2x, 2y, 2z \rangle, \text{ and } \nabla w(1, 1, 1) = \langle 2, 2, 2 \rangle.$$

The tangent plane is

$$2(x - 1) + 2(y - 1) + 2(z - 1) = 0, \text{ or } x + y + z = 3. \ \square$$

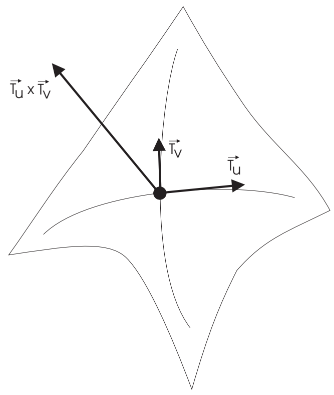
Now suppose the surface is given in parametric form:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

How can I find the normal to the surface at $u = a$, $v = b$?

If I fix $v = b$ and let u vary, I get a curve in the surface. The tangent $\overrightarrow{T_u}$ to this curve will be tangent to the surface.

Likewise, if I fix $u = a$ but let v vary, I get a curve in the surface. The tangent $\overrightarrow{T_v}$ to this curve will be tangent to the surface.



The cross product $\overrightarrow{T_u} \times \overrightarrow{T_v}$ will be perpendicular to the surface.

How do I find $\overrightarrow{T_u}$ and $\overrightarrow{T_v}$? Consider $\overrightarrow{T_u}$. If I fix v and vary u, the tangent vector will be the derivative of the curve

$$\langle f(u,b), g(u,b), h(u,b)\rangle.$$

This is just $\left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u}\right\rangle$, so

$$\overrightarrow{T_u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle.$$

Likewise,

$$\overrightarrow{T_v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle.$$

Hence, the normal vector is the cross product

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

Example. Find the equation of the tangent plane to

$$x = (u^2+u)\cos v, \quad y = (u^2+u)\sin v, \quad z = u^2 - u$$

at $u = 1$, $v = \dfrac{\pi}{2}$.

The point on the surface is $(0,2,0)$ (plug $u = 1$, $v = \dfrac{\pi}{2}$ into x, y, z).

Since I only need the normal at one point, I'll compute $\overrightarrow{T_u}$ and $\overrightarrow{T_v}$ at $u = 1$, $v = \dfrac{\pi}{2}$, *then* take the cross product.

$$\overrightarrow{T_u} = \langle (2u+1)\cos v, (2u+1)\sin v, 2u-1\rangle \text{ and } \overrightarrow{T_v} = \langle -(u^2+u)\sin v, (u^2+u)\cos v, 0\rangle.$$

Therefore,

$$\overrightarrow{T_u}\left(1, \frac{\pi}{2}\right) = \langle 0,3,1\rangle \text{ and } \overrightarrow{T_v}\left(1, \frac{\pi}{2}\right) = \langle -2,0,0\rangle.$$

The normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & 1 \\ -2 & 0 & 0 \end{vmatrix} = \langle 0,-2,6\rangle.$$

The tangent plane is

$$(0)(x-0) + (-2)(y-2) + (6)(z-0) = 0, \text{ or } y - 3z = 2. \ \square$$

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