8-29-1993

Review: Triple integrals.

If R is a region in space, the volume of R is

$$V = \int\int_R\int dx\, dy\, dz.$$

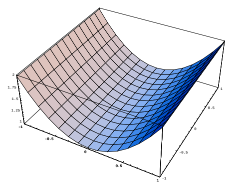
(Of course, you can usually use a double integral to compute the volume of a solid.)
The **centroid** of the region is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\displaystyle\int\int_R\int x\, dx\, dy\, dz}{\displaystyle\int\int_R\int dx\, dy\, dz}, \quad \overline{y} = \frac{\displaystyle\int\int_R\int y\, dx\, dy\, dz}{\displaystyle\int\int_R\int dx\, dy\, dz}, \quad \overline{z} = \frac{\displaystyle\int\int_R\int z\, dx\, dy\, dz}{\displaystyle\int\int_R\int dx\, dy\, dz}.$$

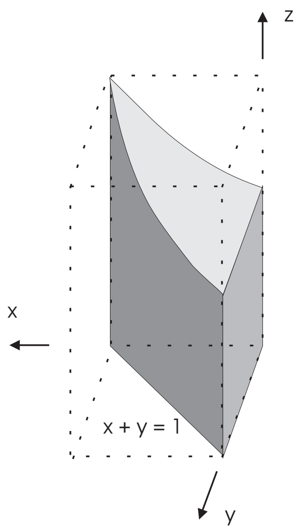
The expressions in the numerators of the fractions are the **x-moment**, the **y-moment**, and the **z-moment**, respectively.
The centroid is just the center of mass for an object with constant density.

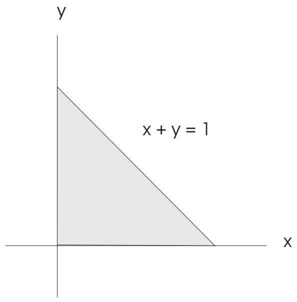
Example. Find the centroid of the region in the first octant bounded by $z = x^2 + 1$ and $x + y = 1$.

$z = x^2 + 1$ is a parabolic trough. It forms the top of the solid.



The plane $x + y = 1$ cuts diagonally across the trough, and forms the front "wall" of the solid. I've sketched the solid in the first picture below.





The projection of the solid into the x-y plane is shown in the second picture. It is described by the inequalities

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

The bottom of the solid is the x-y plane, $z = 0$. The top is $z = x^2 + 1$. Therefore, the limits on z are

$$0 \leq z \leq x^2 + 1.$$

I'll compute the volume first, then the x, y, and z moments. The volume is

$$V = \int_0^1 \int_0^{1-x} (x^2 + 1)\, dz\, dy\, dx = \frac{7}{12}.$$

I used a double integral for the volume, but I need triple integrals for the moments. They are

$$x\text{-moment} = \int_0^1 \int_0^{1-x} \int_0^{x^2+1} x\, dz\, dy\, dx = \frac{13}{60}$$

$$y\text{-moment} = \int_0^1 \int_0^{1-x} \int_0^{x^2+1} y\, dz\, dy\, dx = \frac{11}{60}$$

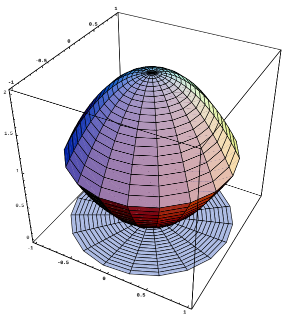
$$z\text{-moment} = \int_0^1 \int_0^{1-x} \int_0^{x^2+1} z\, dz\, dy\, dx = \frac{7}{20}$$

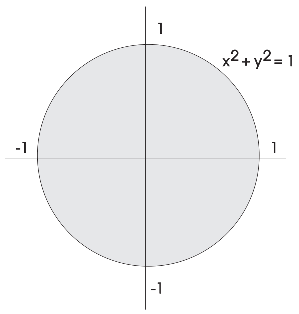
The coordinates of the centroid are

$$\overline{x} = \frac{\frac{13}{60}}{\frac{7}{12}} = \frac{13}{35} \approx 0.37143, \quad \overline{y} = \frac{\frac{11}{60}}{\frac{7}{12}} = \frac{11}{35} \approx 0.31429, \quad \overline{z} = \frac{\frac{7}{20}}{\frac{7}{12}} = \frac{3}{5} = 0.6. \square$$

Example. Let R be the region bounded above by the paraboloid $z = 2 - x^2 - y^2$ and below by the paraboloid $z = x^2 + y^2$. Describe R by inequalities by: (a) Projecting R into the x-y plane; (b) Projecting R into the y-z plane.

The first picture shows the two paraboloids. It appears as though they intersect in a circle.





Set the two paraboloid equations equal: $2 - x^2 - y^2 = x^2 + y^2$ gives $x^2 + y^2 = 1$. They intersect in a circle of radius 1. Therefore, the projection of the solid into the x-y plane is the interior of the circle $x^2 + y^2 = 1$, which I've sketched in the second picture.

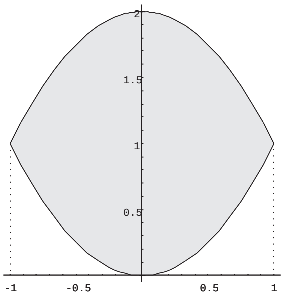
Since the x-y projection is a circle, it's natural to convert to cylindrical coordinates. The projection is

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

The bottom surface is $z = x^2+y^2 = r^2$, and the top surface is $z = 2-x^2-y^2 = 2-r^2$. So $r^2 \le z \le 2-r^2$. The inequalities for the region are

$$\left\{ \begin{array}{c} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ r^2 \le z \le 2 - r^2 \end{array} \right\}$$

Next, I'll project the region into the y-z plane. The y-z plane is $x = 0$; if I set $x = 0$ in the two paraboloid equations, I get $z = 2 - y^2$ and $z = y^2$. Picture:



The parabolas intersect at $y = \pm 1$. It appears that I can use the inequalities

$$-1 \le y \le 1, \quad y^2 \le z \le 2 - y^2.$$

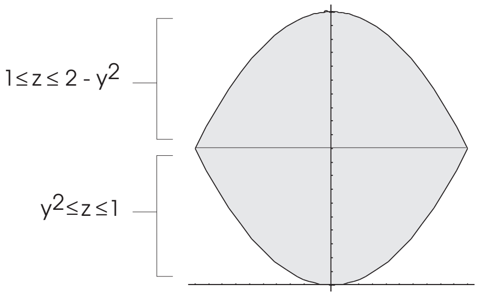
However, it's not so simple. What happens when I try to find the bounds on x? Look at the picture of the two surfaces that I gave at the start. If I look at the lower paraboloid $z = x^2 + y^2$, I find that $x = \pm\sqrt{z - y^2}$. The bounds on x are

$$-\sqrt{z - y^2} \le x \le \sqrt{z - y^2}.$$

If I look at the upper paraboloid $z = 2 - x^2 - y^2$, I find that $x = \pm\sqrt{2 - z - y^2}$. This time, the bounds on x are

$$-\sqrt{2 - z - y^2} \le x \le \sqrt{2 - z - y^2}.$$

In other words, I have different limits for x depending on what part of the region I'm looking at. Now in our projection picture, the lower paraboloid corresponds to $y^2 \le z \le 1$ and the upper paraboloid to $1 \le z \le 2 - y^2$.



So in this case, I need two sets of inequalities to describe the region:

$$\left\{ \begin{array}{c} -1 \le y \le 1 \\ y^2 \le z \le 1 \\ -\sqrt{z - y^2} \le x \le \sqrt{z - y^2} \end{array} \right\} \quad \left\{ \begin{array}{c} -1 \le y \le 1 \\ 1 \le z \le 2 - y^2 \\ -\sqrt{2 - z - y^2} \le x \le \sqrt{2 - z - y^2} \end{array} \right\}$$

In all likelihood you would be better off setting up a triple integral over the region using the x-y plane projection, rather than the y-z plane projection. If you used the y-z plane projection, you would need to set up *two* triple integrals (for the two sets of inequalities) and then add the results. □