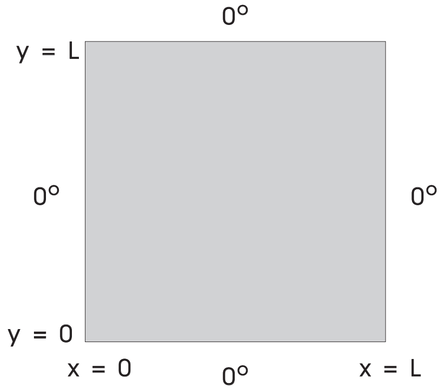
11-15-1993

Heat conduction in two dimensions.

I'll look at a simple situation involving heat conduction in two dimensions. Consider a square plate occupying the region $0 \le x \le L$, $0 \le y \le L$ in the x-y plane.



Assume that no heat is gained or lost through the faces of the plate — heat only moves in or out through the edges. The temperature $u(x, y, t)$ is a function of the time t and the position (x, y) on the plate. It satisfies the two dimensional heat equation

$$u_t = a^2(u_{xx} + u_{yy}), \qquad 0 < x, y < L, \quad t > 0.$$

($u_{xx} + u_{yy}$ is just the **Laplacian** $\nabla^2 u$.)

As indicated in the picture above, I'll consider the case where the edges are always held at $0°$. (You can consider more general situations in which some edges are held at different temperatures, or varying temperatures, or are insulated. To avoid complications, I'll confine myself to this simple case.) The boundary conditions are given by the equations

$$u(0, y, t) = u(L, y, t) = u(x, 0, t) = u(x, L, t) = 0, \qquad 0 < x, y < L, \quad t > 0.$$

I'll also assume that $|u(x, y, t)| \le M$ for some constant M. This means that the temperature does not go to infinity, which certainly conforms to physical intuition.

Solve the equation by separation of variables: Try $u(x, y, t) = f(x)g(y)h(t)$. Then

$$f(x)g(y)h''(t) = a^2\left(f''(x)g(y)h(t) + f(x)g''(y)h(t)\right), \text{ or } \frac{h'(t)}{h(t)} = a^2\left(\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right).$$

The left side is a function of t, while the right side is a function of x and y, so both sides must be constant:

$$\frac{h'(t)}{h(t)} = a^2\left(\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)}\right) = -\lambda^2.$$

The constant really is negative, as I've written it; I'm about the justify that now.

Consider the t part:

$$\frac{h'(t)}{h(t)} = -\lambda^2.$$

This is an exponential decay equation, and the solution is

$$h(t) = be^{-\lambda^2 t}.$$

Now you can see why I used the negative constant $-\lambda^2$ rather than a positive constant λ^2. With a positive constant, this would be a positive exponential, which would go to infinity as $t \to \infty$. But this would violate the boundedness assumption I made above.

Now consider the x-y part of the equation:

$$a^2 \left(\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} \right) = -\lambda^2.$$

Rewrite it as

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} - \frac{\lambda^2}{a^2}.$$

The left side is a function of x, while the right side is a function of y. Therefore, both sides must be constant:

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} - \frac{\lambda^2}{a^2} = -\mu^2.$$

The constant must be negative again. Here's why.

If I use a positive constant μ^2, I have $\dfrac{f''(x)}{f(x)} = \mu^2$, or $f''(x) = \mu^2 f(x)$. This equation is satisfied by $e^{\mu x}$ and $e^{-\mu x}$, as you can check by plugging these in for $f(x)$. So the solution is

$$f(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

But plugging in the boundary conditions $u(0, y, t) = u(L, y, t) = 0$ and solving for the constants gives $c_1 = c_2 = 0$, so $f \equiv 0$, and $u \equiv 0$. This doesn't make sense unless the initial temperature distribution is identically 0, which doesn't make for a very interesting problem. So I will rule this case out.

Continuing with the x-y equation, I have

$$\frac{f''(x)}{f(x)} = -\mu^2.$$

The solution (as in the one dimensional heat equation) is

$$f(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

I also have

$$-\frac{g''(y)}{g(y)} - \frac{\lambda^2}{a^2} = -\mu^2, \text{ or } \frac{g''(y)}{g(y)} = \mu^2 - \frac{\lambda^2}{a^2}.$$

An argument similar to the one involving μ^2 shows that the constant $\mu^2 - \dfrac{\lambda^2}{a^2}$ must be negative. (I'm about to take its square root, so I must be sure to take the square root of $\dfrac{\lambda^2}{a^2} - \mu^2$.) The solution is

$$g(y) = d_1 \cos \left(y \sqrt{\frac{\lambda^2}{a^2} - \mu^2} \right) + d_2 \sin \left(y \sqrt{\frac{\lambda^2}{a^2} - \mu^2} \right).$$

Yuck.

Remember that $u(x, y, t) = f(x)g(y)h(t)$. I'm going to plug in the boundary conditions to find some of the constants.

First,

$$0 = u(0, y, t) = f(0)g(y)h(t), \quad f(0) = 0, \quad c_1 = 0.$$

(I can cancel some stuff here, because the equation $0 = f(0)g(y)h(t)$ is an identity. If $g(y) \equiv 0$ or $h(t) \equiv 0$, I get $u \equiv 0$, which is the stupid case I ruled out above.) Thus,

$$f(x) = c_2 \sin \mu x.$$

Next,

$$0 = u(L, y, t) = f(L)g(y)h(t), \quad f(L) = 0, \quad c_2 \sin \mu L = 0.$$

If $c_2 = 0$, $f \equiv 0$ and $u \equiv 0$ (the stupid case again), so I have $\sin \mu L = 0$. Then μL must be a multiple of π: $\mu L = n\pi$, and $\mu = \dfrac{\pi n}{L}$. Hence,

$$f(x) = c_2 \sin \frac{\pi n x}{L}.$$

(Notice that this is the same form as in the one dimensional heat equation!)

Now I have to plug in the boundary conditions for y. First,

$$0 = u(x, 0, t) = f(x)g(0)h(t), \quad g(0) = 0, \quad d_1 = 0.$$

Thus,

$$g(y) = d_2 \sin \left(y \sqrt{\frac{\lambda^2}{a^2} - \mu^2} \right).$$

Now $\mu = \dfrac{\pi n}{L}$, so

$$g(y) = d_2 \sin \left(y \sqrt{\frac{\lambda^2}{a^2} - \frac{\pi^2 n^2}{L^2}} \right).$$

(One more plug in to go!) Next,

$$0 = u(x, L, t) = f(x)g(L)h(t), \quad g(L) = 0, \quad d_2 \sin \left(L \sqrt{\frac{\lambda^2}{a^2} - \frac{\pi^2 n^2}{L^2}} \right).$$

To avoid $d_2 = 0$ (which gives the stupid solution again), the ugly stuff inside the sine must be a multiple of pi:

$$L \sqrt{\frac{\lambda^2}{a^2} - \frac{\pi^2 n^2}{L^2}} = \pi m, \quad , \sqrt{\frac{\lambda^2 L^2}{a^2} - \pi^2 n^2} = \pi m, \quad \lambda^2 = \frac{\pi^2 a^2}{L^2}(m^2 + n^2).$$

Now plug this back into $g(y)$ and $h(t)$, where λ occurs. After simplifying a bit,

$$g(y) = d_2 \sin \frac{\pi m y}{L} \quad \text{and} \quad h(t) = c \exp \left(-\frac{\pi^2 a^2 n^2 t}{L^2} \right).$$

And hence,

$$u(x, y, t) = k \exp \left(-\frac{\pi^2 a^2 n^2 t}{L^2} \right) \sin \frac{\pi n x}{L} \sin \frac{\pi m y}{L}.$$

(I've set $k = c \cdot c_2 \cdot d_2$.) Notice that I have two integer parameters m and n. When I superpose to get the final result, I must sum on m and on n:

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{m,n} \exp \left(-\frac{\pi^2 a^2 n^2 t}{L^2} \right) \sin \frac{\pi n x}{L} \sin \frac{\pi m y}{L}.$$

As the notation indicates, the coefficients $B_{m,n}$ are Fourier sine coefficients of a kind. In fact, they are coefficients from a **double Fourier sine series**, and are given by

$$B_{m,n} = \frac{4}{L^2} \int_0^L \int_0^L u(x, y, 0) \sin \frac{\pi n x}{L} \sin \frac{\pi m y}{L} \, dx \, dy.$$

Compare this to the formula for the ordinary Fourier sine coefficients and you'll see that this is reasonable. $u(x, y, 0)$ is the initial temperature distribution, and you compute the coefficients by computing the double integral.

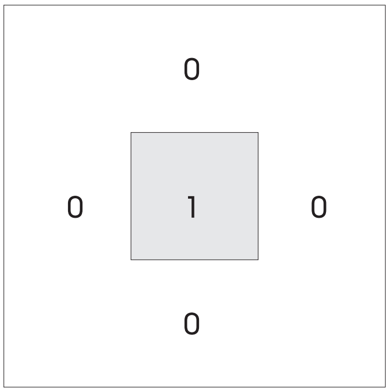
Example. Solve the two dimensional heat conduction problem

$$u_t = u_{xx} + u_{yy}, \qquad 0 < x, y < 1, \quad t > 0$$

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0, \qquad 0 < x, y < 1, \quad t > 0$$

$$u(x, y, 0) = \begin{cases} 1 & \text{if } \dfrac{1}{3} \le x, y \le \dfrac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

Here's a picture of the initial temperature distribution:



The double Fourier sine coefficients are

$$B_{m,n} = \frac{4}{1} \int_0^1 \int_0^1 u(x, y, 0) \sin(\pi n x) \sin(\pi m y)\, dx\, dy = 4 \int_{1/3}^{2/3} \int_{1/3}^{2/3} \sin(\pi n x) \sin(\pi m y)\, dx\, dy =$$

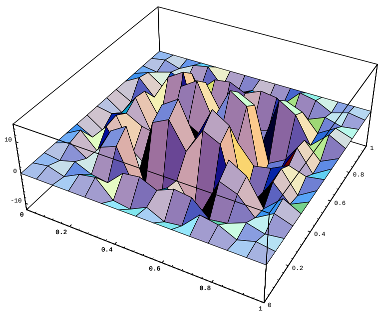
$$\frac{4}{\pi^2 mn} \left(\cos \frac{2\pi n}{3} - \cos \frac{\pi n}{3} \right) \left(\cos \frac{2\pi m}{3} - \cos \frac{\pi m}{3} \right).$$

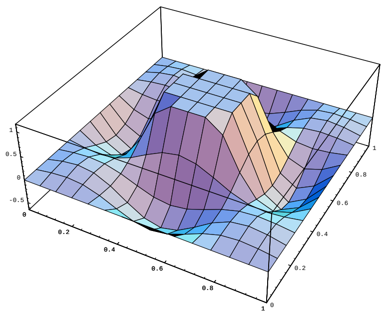
The solution is

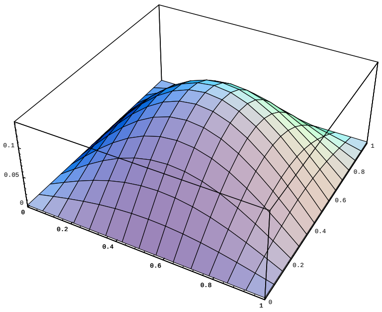
$$u(x, y, t) =$$

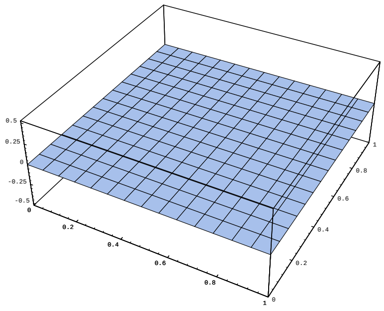
$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{\pi^2 mn} \left(\cos \frac{2\pi n}{3} - \cos \frac{\pi n}{3} \right) \left(\cos \frac{2\pi m}{3} - \cos \frac{\pi m}{3} \right) \exp\left(-\pi^2 (m^2 + n^2) t\right) \sin(\pi n x) \sin(\pi m y).$$

I used the partial sum of the terms $n = 1, \ldots, 10$, $m = 1, \ldots, 10$ to approximate the solution $u(x, y, t)$, then plotted the (approximate) solution for $t = 0$, $t = 0.01$, $t = 0.1$, and $t = 1.75$.









The first plot should be a square-shaped plateau; it doesn't look like one because of the error involved in using the partial sum instead of the actual solution. As t increases, the heat diffuses away from the plateau toward the edges, which are held at $0°$. By $t = 1.75$, the temperature is close to 0 over the whole square. $\square$