Math 227
9-13-1993

Parametrizing surfaces.

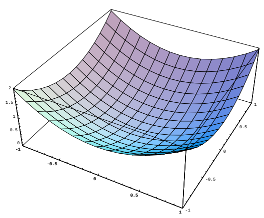
Name: ______________________________________

This project is due by 3 p.m., Monday, September 27.

1. What does it mean to parametrize a surface?

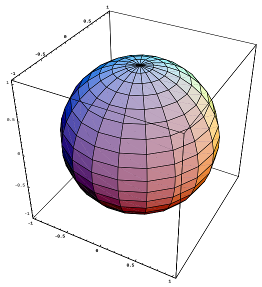
The graph of a function $z = f(x, y)$ is a surface in $\mathbb{R}^3$. For example, the graph of $z = x^2 + y^2$ is a paraboloid.

```
Plot3D[x^2 + y^2, {x, -1, 1}, {y, -1, 1}]
```



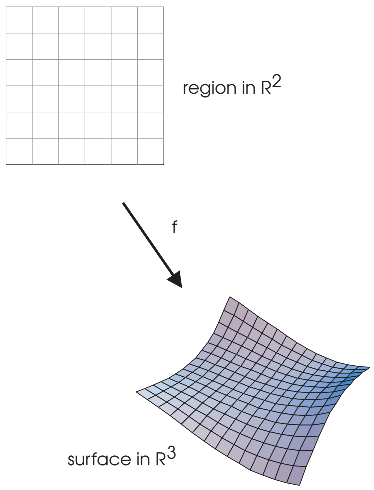
On the other hand, a sphere is not the graph of a function:

```
ParametricPlot3D[{Sin[p] Cos[t], Sin[p] Sin[t], Cos[p]},
    {p, 0, Pi}, {t, 0, 2 Pi}]
```



This difference is reflected in the different commands used to plot the two surfaces. A surface which is the graph of a function can be plotted with `Plot3D`; a **parametrized surface** can be plotted with `ParametricPlot3D`. (A surface which is the graph of a function can be represented as a parametrized surface, so the second notion is therefore more general than the first.)

A **parametrization** of a surface is a function $f : \mathbb{R}^2 \to \mathbb{R}^3$. The inputs to f are points in the plane; the outputs are points in space. You can think of f as "deforming" the plane $\mathbb{R}^2$ to produce a surface.



The surface is *not* the graph of f. The graph of f would be an object in 5-dimensional space — to graph a function, you need one coordinate axis for each input variable and each output variable.

Sometimes, a parametrization will be given in component form. If the input variables to f are u and v and the output variables are x, y, and z, then x, y, and z are functions of u and v:

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v).$$

For example, the parametrization of the sphere I used the the `ParametricPlot3D` statement above was

$$x = \sin p \cos t, \quad y = \sin p \sin t, \quad z = \cos p.$$

Compare it to the spherical coordinate conversion equations and you'll see where it comes from.

The examples and problems that follow involve a lot of pictures in 3 dimensions. These pictures eat up huge amounts of memory, and this can cause *Mathematica* to become unstable. Save your work after each picture to be on the safe side!

2. How do you parametrize a surface?

There is no general method for parametrizing a surface. Here are some important cases.

a. The surface is given by an equation which can be solved for one of the variables.

Example. $z = x^2 - y^2$

Set the independent variables equal to the two parameters:

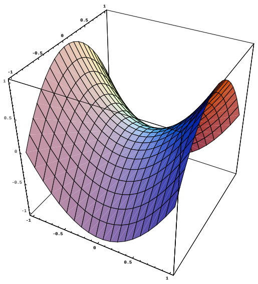
$$x = u, \quad y = v.$$

Next, use the given equation to parametrize the dependent variable:

$$z = x^2 - y^2 = u^2 - v^2.$$

Check the parametrization by drawing the picture:

```
ParametricPlot3D[{u, v, u^2 - v^2}, {u, -1, 1}, {v, -1, 1}]
```



How do you know what ranges of values to use for the parameters `u` and `v` when you plot the surface? Well, most of the time, it's a matter of trial and error. You should avoid values of the parameters which would cause arithmetic errors, such as division by zero. You will also get nicer pictures if the range of inputs is roughly the same size as the range of outputs. $\square$

1. (a) Parametrize the surface $z = 4 - x^2 - y^2$ by setting x and y equal to the two parameters u and v.

(b) Use `ParametricPlot3D` to plot the graph. Make a rough sketch of the graph in the box below. (Suggestion: Use $-2 \leq u \leq 2$, $-2 \leq v \leq 2$.)

Here is another way to parametrize the surface which uses the *circular symmetry* — notice the $x^2 + y^2$ combination?

2. (a) Let $z = 4 - x^2 - y^2$. Express x, y, and z in terms of the polar coordinate variables r and θ.

(b) The equations you wrote down in part (a) parametrize the surface $z = 4-x^2-y^2$. Use this parametrization and the `ParametricPlot3D` function to draw the surface. Make a rough sketch of the graph in the box below. (Suggestion: Use t instead of θ. Try the ranges $0 \le r \le 2$ and $0 \le t \le 2\pi$.)

b. The surface is given by an equation involving only two variables.

Example. $x^2 - y^2 = 1$, $x > 0$.

First, set the "unrestricted" variable equal to one of the parameters: $z = v$.

Next, think of the given equation as the equation of a curve. Use the remaining parameter to parametrize the curve.

A useful parametrization of $x^2 - y^2 = 1$ is

$$x = \sec u, \quad y = \tan u.$$

This works, because $(\sec u)^2 - (\tan u)^2 = 1$.

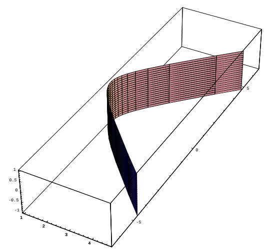
3. Suppose I use $x = \sec u$, $y = \tan u$, to parametrize the hyperbola $x^2 - y^2 = 1$, and I use the parameter range $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$. Do I get the whole hyperbola? If not, why not?

The surface may be parametrized by

$$x = \sec u, \quad y = \tan u, \quad z = v.$$

Check the result by drawing the surface:

```
ParametricPlot3D[{Sec[u], Tan[u], v}, {u, -Pi/2+0.1, Pi/2-0.1}, {v, -1, 1}]
```



□

4. (a) Show by substitution that

$$x = 1 - 3u^2, \quad y = u(3 - u^2),$$

parametrizes the *curve* $27y^2 = (1 - x)(x + 8)^2$ (Tschirnhausen's cubic).

To do this with *Mathematica*, substitute the u stuff for x and y in the left and right sides and check that you get the same thing. For example, here's how to do the left side:

```
Expand[27y^2 /.  (y->u(3 - u^2))]
```

Now expand the right side, and *write down* what you get as confirmation. (You can do this by hand if you're energetic.)

(b) Parametrize the *surface* $27y^2 = (1 - x)(x + 8)^2$. Be sure you get x, y, and z in terms of u *and* v!

(c) Use `ParametricPlot3D` to plot the *surface* $27y^2 = (1 - x)(x + 8)^2$. Make a rough sketch of the surface in the box below. (I suggest a range of -2 to 2 for u and v.)

c. A parametrization related to a change-of-variables.

The polar conversion equations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

give a function from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$(x, y) = f(r, \theta) = (r\cos\theta, r\sin\theta).$$

If $z = g(r, \theta)$ (some junk involving r and θ), you get a function from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$(x, y, z) = F(r, \theta) = (r\cos\theta, r\sin\theta, g(r, \theta)).$$

This is a parametrized surface in $\mathbb{R}^3$.

For example, suppose $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. The equation

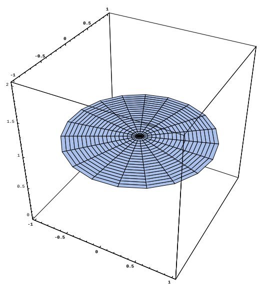
$$(x, y) = f(r, \theta) = (r\cos\theta, r\sin\theta)$$

produces the standard disk of radius 1 centered at the origin of the x-y plane. Setting $z = 1$ moves the disk up one unit so that it lies in the plane $z = 1$:

$$(x, y, z) = F(r, \theta) = (r\cos\theta, r\sin\theta, 1).$$

I'll check by plotting it. (I'm naming the picture `disk` because I want to use it again below.)

```
disk = ParametricPlot3D[{r Cos[t], r Sin[t], 1}, {r, 0, 1}, {t, 0, 2 Pi}]
```



$\square$

5. The circle $x^2 + y^2 = 2x$ can be parametrized by

$$x = \cos u + 1, \qquad y = \sin u,$$

for $0 \le u \le 2\pi$. Use this to parametrize the *cylinder* $x^2 + y^2 = 2x$ for $0 \le z \le 1$. (Refer to the discussion of parametrizing a surface given by an equation in two variables if you get stuck.) Write down your parametrization below.

Use your parametrization to fill in the "`...`" in the command below. Enter the command yourself and verify that you get a cylinder. (The "`cylinder =`" is there so I can refer to the picture below.)

```
cylinder = ParametricPlot3D[ {...}, {u, 0, 2 Pi}, {v, 0, 1}]
```

6. *Mathematica* has the ability to combine graphics objects, which makes it easy to put together complicated pictures from their pieces. Try the following:

```
Show[disk, cylinder]
```

Make a rough sketch of the result in the box below.

d. (Surfaces of revolution) The surface is generated by revolving a curve about a line.

The line about which the curve is revolved is called the **axis of revolution**. For simplicity, assume that the line and the curve lie in one of the coordinate planes, and that the line is one of the coordinate axes.

Let w_1, w_2, and w_3 be the coordinate axes. (I'm using these names instead of "x", "y", and "z" so that you don't get confused when you translate this description to an actual problem.) Suppose the curve is in the w_1-w_2 plane, and is given parametrically by

$$w_1 = f(u), \quad w_2 = g(u).$$

Suppose the curve is being revolved about the w_1-axis. The surface of revolution can be parametrized by

$$w_1 = f(u), \quad w_2 = g(u)\sin v, \quad w_3 = g(u)\cos v.$$

You can see this is right if you draw a careful picture. But the given equations surely make sense: w_1 (the axis) remains "fixed" when the curve is revolved about it, so it's the same as w_1 for the curve. The *revolution* of the curve comes from the standard *circular* parametrization involving sine and cosine.

Example. I'll parametrize the surface generated by revolving the circle

$$(y-2)^2 + (z-2)^2 = 1$$

about the z-axis.

First, parametrize the circle:

$$y = 2 + \cos u, \quad z = 2 + \sin u.$$

(You can check that it works by substituting for x and y in the circle equation.)

In terms of the notation above, y corresponds to w_2 and z corresponds to w_1. Hence, x corresponds to w_3. Thus,

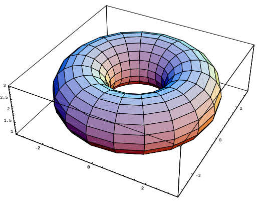
$$f(u) = 2 + \sin u, \quad g(u) = 2 + \cos u.$$

The parametrization is

$$x = (2 + \cos u)\cos v, \quad y = (2 + \cos u)\sin v, \quad z = 2 + \sin u.$$

Check the result by drawing the surface:

```
ParametricPlot3D[{(2 + Cos[u]) Cos[v], (2 + Cos[u]) Sin[v], 2 + Sin[u]},
    {u, 0, 2 Pi }, {v, 0, 2 Pi}]
```



Revolving the circle about the z-axis produces a **torus of revolution**. □

7. (a) `Plot` the curve $y = x(1-x)$ for $0 \le x \le 2$. Make a rough sketch of the graph in the box below.

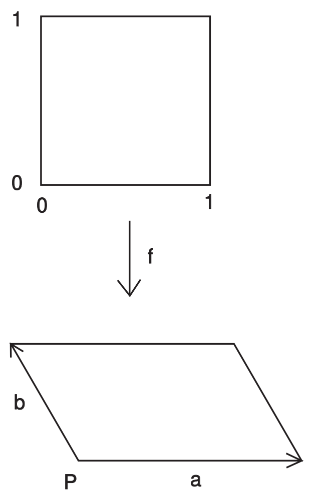
(b) Parametrize the *curve* $y = x(1-x)$. (Hint: Begin by setting $x = u$.)

(c) Parametrize the surface of revolution obtained by revolving the curve $y = x(1-x)$ for $0 \le x \le 2$, about the x-axis.

(d) Use `ParametricPlot3D` to plot the surface obtained by revolving the curve $y = x(1 - x)$, $0 \le x \le 2$, about the x-axis. Make a rough sketch of the surface in the box below. (Suggestion: If you used u to parametrize the curve in part (b), try the ranges $0 \le u \le 2$ and $0 \le v \le 2\pi$.)

e. Parallelograms in $\mathbb{R}^3$.

Suppose one corner of the paralleogram is the point $P(p_1, p_2, p_3)$ and the sides emanating from P are given by the vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$.



I'll construct a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which maps the unit square $0 \le x \le 1$, $0 \le y \le 1$ to the parallelogram. (It's essentially the same as the affine coordinate transformation which maps the square to a parallelogram in $\mathbb{R}^2$. The only difference is that the target is in $\mathbb{R}^3$.)

The sides of the unit square are given by the vectors $\hat{\imath} = \langle 1, 0\rangle$, $\hat{\jmath} = \langle 0, 1\rangle$. First, I'll build a 3×2 matrix M such that

$$M \cdot \hat{\imath} = \vec{a}, \quad M \cdot \hat{\jmath} = \vec{b}.$$

If

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix},$$

then

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Multiply out the left sides:

$$\begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Therefore,

$$M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

That is, the columns of M are just $\vec{a}$ and $\vec{b}$.

Now multiplication by M will carry $\hat{\imath}$ to $\vec{a}$ and $\hat{\jmath}$ to $\vec{b}$, so it will carry the square to the parallelogram ... almost. The "$\vec{a}$" and "$\vec{b}$" produced will be based at the origin $(0, 0, 0)$, whereas the target parallelogram has them based at the point P.

To fix this, add P to translate everything from $(0, 0, 0)$ to P. Here's the result:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}. \square$$

Example. Consider the parallelogram with vertices $P(1, 1, 2)$, $Q(2, 2, 3)$, $R(3, 1, 1)$, and $S(4, 2, 2)$. The vectors starting at P are

$$\overrightarrow{PQ} = \langle 1, 1, 1 \rangle, \quad \overrightarrow{PR} = \langle 2, 0, -1 \rangle.$$

The parametrization is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

You can write this in component form by multiplying the matrices:

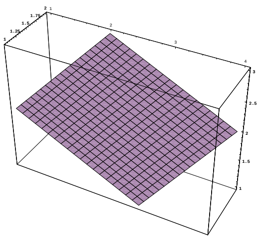
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u + 2v + 1 \\ u + 1 \\ u - v + 2 \end{bmatrix}.$$

That is,

$$x = u + 2v + 1, \quad y = u + 1, \quad z = u - v + 2.$$

Check the result by drawing the surface:

```
ParametricPlot3D[{u + 2v + 1, u + 1, u - v + 2}, {u, 0, 1 }, {v, 0, 1}]
```



$\square$

8. (a) The point $P(1,3,0)$ is one corner of a parallelogram. The sides emanating from P are given by the vectors $\langle 2,4,1\rangle$ and $\langle 1,-3,2\rangle$. Parametrize the parallelogram.

(b) Use `ParametricPlot3D` to plot the parallelogram using the parametrization you found in (a). Draw a rough sketch of the picture in the box below.

f. Segment parametrizations.

The trick I'll illustrate in the next example works when you can decompose the surface (or its projection into a coordinate plane) into segments. Before I begin, let me remind you that if P and Q are points, the

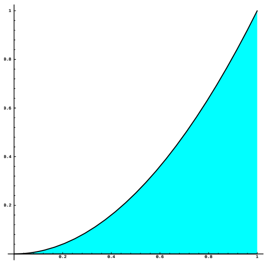
line segment from P to Q may be parametrized by

$$(x, y, z) = (1-u)\cdot P + u\cdot Q, \quad 0 \le u \le 1.$$

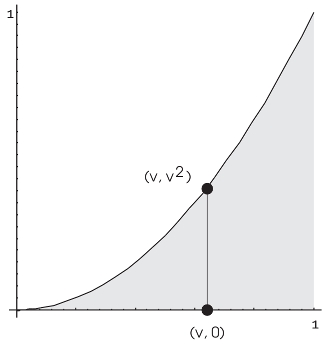
The same thing works in 2 dimensions — just write (x, y) on the left.

Example. I'll parametrize the part of $z = x^2 + y^2$ which lies above the region

$$0 \le x \le 1, \quad 0 \le y \le x^2.$$



The projection of the surface into the x-y plane is the given region $0 \le x \le 1$, $0 \le y \le x^2$. Decompose it into vertical segments. I've drawn the segment at $x = v$ below:



The bottom point is $(v, 0)$ and the top point is (v, v^2), so the segment joining the points is

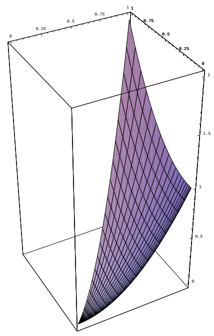
$$(x, y) = (1-u)(v, 0) + u(v, v^2) = ((1-u)v, 0) + (uv, uv^2) = (u, uv^2).$$

Since v is x, I need $0 \le v \le 1$, and $0 \le u \le 1$ for the segment parametrization. Now that I have x and y in terms of u and v, I just plug in to get z:

$$z = x^2 + y^2 = u^2 + u^2v^4.$$

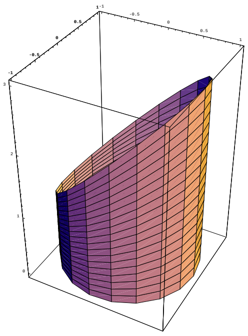
Check by drawing it:

```
ParametricPlot3D[{u, u v^2, u^2 + u^2 v^4},
    {u, 0, 1}, {v, 0, 1}, ViewPoint->{-1.3, -2.4, 2.0}]
```



(I changed `ViewPoint` from the default to get a better picture.) □

9. Consider the surface which consists of the part of the cylinder $x^2 + y^2 = 1$ satisfying $0 \le z \le x + 2$. Parametrize it by decomposing it into vertical segments (write down the parametrization below).



(Hint: If a point on the bottom circle is $(\cos v, \sin v)$, the point on the top edge directly above it is $(\cos v, \sin v, \cos v + 2)$.)