3-6-1994

The convolution formula.

The **convolution** of f and g is

$$(f \star g)(t) = \int_0^t f(u)g(t-u)\,du.$$

The transform of a convolution is given by

$$\mathcal{L}\left((f \star g)(t)\right) = F(s)G(s),$$

where as usual F and G are the transforms of f and g, respectively.

This is very useful going the other way:

$$\mathcal{L}^{-1}\left[F(s)G(s)\right] = (f \star g)(t).$$

The idea is that if I'm trying to compute the inverse transform of $Junk$ and $Junk$ is a product $F(s)G(s)$ of two things whose inverse transforms I know, then I can compute the inverse transform of $Junk$ by taking the convolution of the inverse transforms of its factors.

First, what is the convolution of two functions? I'll do a computational example, then illustrate the idea with pictures.

Example. The convolution of $f(t) = t$ and $g(t) = e^t$ is

$$t \star e^t = \int_0^t ue^{t-u}\,du = \int_0^t ue^t e^{-u}\,du = e^t \int_0^t ue^{-u}\,du.$$

Do the integral by parts:

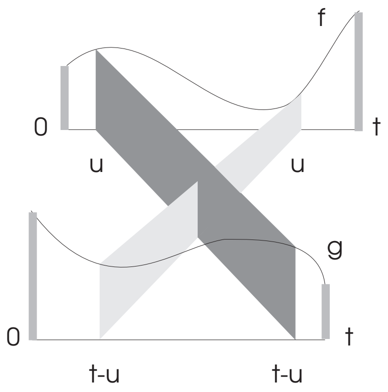
$$\begin{array}{cccc} & \dfrac{d}{du} & & \int du \\ + & u & & e^{-u} \\ & & \searrow & \\ - & 1 & & -e^{-u} \\ & & \searrow & \\ + & 0 & \rightarrow & e^{-u} \end{array}$$

$$\int_0^t ue^{-u}\,du = \left[-ue^{-u} - e^{-u}\right]_0^t = -te^{-t} - e^{-t} + 1.$$

So

$$t \star e^t = e^t\left(-te^{-t} - e^{-t} + 1\right) = -t - 1 + e^t. \ \square$$

Example. The convolution integral $\displaystyle\int_0^t f(u)g(t-u)\,du$ sums the product of f and g from "opposite sides" of the interval $0 \leq u \leq t$:

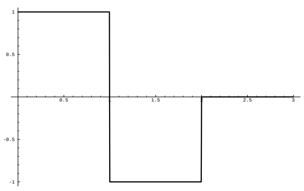


For this reason, convolution is commutative:

$$f \star g = g \star f.$$

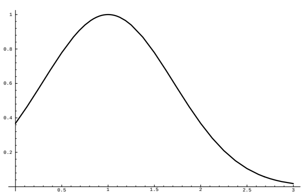
However, when you encounter a convolution in practice, it's likely that the two functions will have different meanings. Here is $f(t)$, a positive pulse followed by a negative pulse:

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ -1 & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases}$$

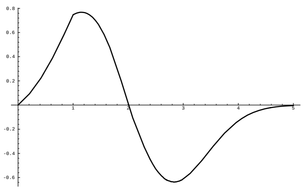


For $g(t)$, I'll use a smooth "bump function" that dies exponentially on either side:

$$g(t) = e^{-(t-1)^2}.$$



Now here is the graph of the convolution $f \star g$.



Notice how the bump function has smeared the sharp transitions in the incoming signal. □

Example.

$$1 \star t = \int_0^t (1)(t-u)\,du = \left[tu - \frac{1}{2}u^2\right]_0^t = \frac{1}{2}t^2.$$

Thus, $1 \star t \neq t$! □

Now let me show you where the convolution transform formula comes from.

$$F(s)G(s) = \left(\int_0^\infty e^{-st} f(t)\,dt\right)\left(\int_0^\infty e^{-sr} g(r)\,dr\right) = \int_0^\infty \int_0^\infty f(t)g(r)e^{-s(r+t)}\,dt\,dr =$$

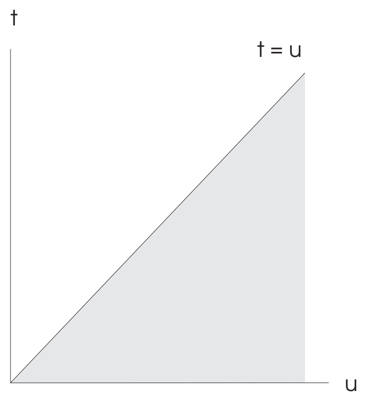
$$\int_0^\infty f(t) \int_0^\infty g(r)e^{-s(r+t)}\,dr\,dt.$$

I'll change variables : Let $u = r + t$, so $du = dr$ (t is constant in the inner integral). $r = 0$ gives $u = t$; as $r \to \infty$, $u \to \infty$. The integral becomes

$$\int_0^\infty f(t) \int_t^\infty s^{-su} g(u-t)\,du\,dt.$$

The region of integration is

$$t \le u < \infty, \quad 0 \le t < \infty.$$



Interchange the order of integration. The new limits are

$$0 \le u < \infty, \quad 0 \le t \le u.$$

The integral becomes

$$\int_0^\infty e^{-su} \int_0^u f(t)g(u-t)\, dt\, du = \int_0^\infty e^{-su}(f \star g)(u)\, du = \mathcal{L}(f \star g)(s).$$

Thus,

$$\mathcal{L}(f \star g) = F(s)G(s).$$

The inverse transform is

$$\mathcal{L}^{-1}(F(s)G(s)) = (f \star g)(t).$$

As I noted above, *you can use this formula to compute the inverse transform of a product.* It is often easier to do the convolution integral than to decompose the product using partial fractions.

Example. $f \star g = g \star f$. For

$$(f \star g)(t) = \int_0^t f(u)g(t-u)\, du.$$

Let $v = t - u$, so $dv = -du$. When $u = 0$, $v = t$, and when $u = t$, $v = 0$:

$$\int_0^t f(u)g(t-u)\, du = \int_t^0 f(t-v)g(v)(-dv) = \int_0^t g(v)f(t-v)\, dv = (g \star f)(t). \ \square$$

Example.

$$t \star \cos t = \int_0^t (t-u)\cos u\, du = \left[(t-u)\sin u - \cos u\right]_0^t = -\cos t + 1.$$

Here is the work for the integral:

$$\begin{array}{ccccc} & \dfrac{d}{du} & & \int du \\ + & t-u & & \cos u \\ & & \searrow & \\ - & -1 & & \sin u \\ & & \searrow & \\ + & 0 & & -\cos u \end{array}$$

$$\int (t-u)\cos u\, du = (t-u)\sin u - \cos u + C. \ \square$$

Example. Compute $\mathcal{L}^{-1}\left(\dfrac{s}{(s^2+1)(s^2+4)}\right)$.

One way to do this is to decompose $\dfrac{s}{(s^2+1)(s^2+4)}$ using partial fractions.

To do this using convolutions, think of $\dfrac{s}{(s^2+1)(s^2+4)}$ as a product:

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1} \cdot \frac{s}{s^2+4}\right).$$

(You can put the "s" on either fraction.)
Now
$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t, \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t.$$

Since I know the inverse transform of each factor, it is reasonable to try to apply the convolution formula. The inverse transform of the product should be the convolution of the inverse transforms of the factors:

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right) = \int_0^t \sin(t-u)\cos 2u\,du.$$

Apply the trig identity
$$\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$$
with $a = t-u$ and $b = 2u$. Then
$$\sin(t-u)\cos 2u = \frac{1}{2}\sin(t+u) + \frac{1}{2}\sin(t-3u).$$

So
$$\int_0^t \sin(t-u)\cos 2u\,du = \int_0^t \left(\frac{1}{2}\sin(t+u) + \frac{1}{2}\sin(t-3u)\right)\,du =$$
$$\left[-\frac{1}{2}\cos(t+u) + \frac{1}{6}\cos(t-3u)\right]_0^t = -\frac{1}{2}\cos 2t + \frac{1}{6}\cos(-2t) + \frac{1}{2}\cos t - \frac{1}{6}\cos t = -\frac{1}{3}\cos 2t + \frac{1}{3}\cos t. \square$$

Example. $y'' - 4y' + 4y = g(t)$, $y(0) = 1$, $y'(0) = -1$.

In this problem, g is not given. I'll obtain a formula for the solution in terms of g.
Transform both sides, writing $G(s)$ for the transform of $g(t)$:

$$\left(1 - s + s^2Y(s)\right) - 4\left(-1 + sY(s)\right) + 4Y(s) = G(s).$$

Solve for $Y(s)$:
$$Y(s) = \frac{s-5}{(s-2)^2} + \frac{G(s)}{(s-2)^2}.$$

For the first term, write
$$\frac{s-5}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{3}{(s-2)^2} = \frac{1}{s-2} - \frac{3}{(s-2)^2}.$$

Then
$$\mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{3}{(s-2)^2}\right) = e^{2t} - 3te^{2t}.$$

The second term $\dfrac{G(s)}{(s-2)^2}$ is the product of two transforms. The inverse transform of $G(s)$ is $g(t)$. The inverse transform of $\dfrac{1}{(s-2)^2}$ is te^{2t}. Hence, by the convolution formula,

$$\mathcal{L}^{-1}\left(\frac{G(s)}{(s-2)^2}\right) = \int_0^t ue^{2u}g(t-u)\,du.$$

The solution is
$$y(t) = e^{2t} - 3te^{2t} + \int_0^t ue^{2u}g(t-u)\,du. \square$$

Example. Solve the **integral equation**

$$y' = 1 + \int_0^t 2y(t-u)e^{-u}\,du, \qquad y(0) = 4.$$

Observe that

$$\int_0^t 2y(t-u)e^{-u}\,du = y \star (2e^t).$$

Therefore, the equation may be written as

$$y' = 1 + y \star (2e^t).$$

Transform both sides:

$$(-4 + sY(s)) = \frac{1}{s} + Y(s) \cdot \frac{2}{s+1}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s+1}{s(s+2)(s-1)} + \frac{4}{(s+2)(s-1)} = \frac{4s^2+5s+1}{s(s-1)(s+2)}.$$

Decompose the right side using partial fractions:

$$\frac{4s^2+5s+1}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2},$$

$$4s^2 + 5s + 1 = A(s-1)(s+2) + Bs(s+2) + Cs(s-1).$$

Setting $s = 0$ gives $1 = -2A$, so $A = -\frac{1}{2}$. Setting $s = 1$ gives $10 = 3B$, so $B = \frac{10}{3}$. Setting $s = -2$ gives $7 = 6C$, so $C = \frac{7}{6}$.

Thus,

$$Y(s) = \frac{4s^2+5s+1}{s(s-1)(s+2)} = -\frac{1}{2}\frac{1}{s} + \frac{10}{3}\frac{1}{s-1} + \frac{7}{6}\frac{1}{s+2}.$$

The solution is

$$y(t) = \mathcal{L}^{-1}\left(-\frac{1}{2}\frac{1}{s} + \frac{10}{3}\frac{1}{s-1} + \frac{7}{6}\frac{1}{s+2}\right) = -\frac{1}{2} + \frac{10}{3}e^t + \frac{7}{6}e^{-2t}. \quad \square$$