4-24-1994

Review for Test 3 (Class).

1. Consider the system

$$\vec{x}\,' = \begin{bmatrix} -2 & 2 \\ 4 & 5 \end{bmatrix} \vec{x}.$$

(a) Solve the system using eigenvectors.

The characteristic polynomial is $\lambda^2 - 3\lambda - 18$. The eigenvalues are $\lambda = 6$, $\lambda = -3$.
Consider $\lambda = 6$:

$$A - 6I = \begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix}$$

An eigenvector $\langle a, b\rangle$ should satisfy

$$\begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two rows of the coefficient matrix are multiples. Throw out the first row; I want to solve

$$4 \cdot a + (-1) \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, 4\rangle$.
Now consider $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

Following the procedure I used for the first eigenvalue, I can throw out the second row. I then need to solve

$$1 \cdot a + 2 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 2, -1\rangle$.
The solution is

$$\vec{x} = c_1 e^{6t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \ \square$$

(b) Solve the system using diagonalization.

Construct a matrix using the eigenvectors from the first part as the columns:

$$S = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}.$$

The inverse is

$$S^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}.$$

Hence, the solution is

$$\vec{x} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \cdot \frac{1}{9} \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \ \square$$

(c) Solve the system using algebraic elimination.

Since the eigenvalues are $\lambda = 6$, $\lambda = -3$,

$$x = c_1 e^{6t} + c_2 e^{-3t}.$$

Then

$$y = \frac{1}{2}(x' + 2x) = \frac{1}{2}\left((6c_1 e^{6t} - 3c_2 e^{-3t}) + 2(c_1 e^{6t} + c_2 e^{-3t})\right) = 4c_1 e^{6t} - \frac{1}{2}c_2 e^{-3t}. \ \square$$

(d) Solve the system using the matrix exponential e^{tA}.

The eigenvalues are $\{6, -3\}$. First, compute the a_k's:

$$a_1 = e^{6t}, \quad a_2 = e^{-3t} \star a_1(t) = \int_0^t e^{-3(t-u)} e^{6u}\, du = \frac{1}{9}(e^{6t} - e^{-3t}).$$

Next, compute the B_k's:

$$B_1 = I, \quad B_2 = A - 6I = \begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix}.$$

So

$$e^{tA} = e^{6t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{9}(e^{6t} - e^{-3t})\begin{bmatrix} -8 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{9}e^{6t} + \frac{8}{9}e^{-3t} & \frac{2}{9}e^{6t} - \frac{2}{9}e^{-3t} \\ \frac{4}{9}e^{6t} - \frac{4}{9}e^{-3t} & \frac{8}{9}e^{6t} + \frac{1}{9}e^{-3t} \end{bmatrix}.$$

The solution is

$$\vec{x} = \begin{bmatrix} \frac{1}{9}e^{6t} + \frac{8}{9}e^{-3t} & \frac{2}{9}e^{6t} - \frac{2}{9}e^{-3t} \\ \frac{4}{9}e^{6t} - \frac{4}{9}e^{-3t} & \frac{8}{9}e^{6t} + \frac{1}{9}e^{-3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_1 \end{bmatrix}. \ \square$$

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2. Solve the system

$$\begin{aligned} x' &= 5x - 5y, \\ y' &= 5x - y. \end{aligned}$$

The eigenvalues are $\lambda = 2 \pm 4i$. I'll find an eigenvector for $2 + 4i$:

$$A - (2+4i)I = \begin{bmatrix} 3 - 4i & -5 \\ 5 & -3 - 4i \end{bmatrix}$$

Throw out the second row. I need to solve

$$(3 - 4i) \cdot a + (-5) \cdot b = 0.$$

An obvious solution is $\langle a, b \rangle = \langle 5, 3 - 4i \rangle$.
The solution for this eigenvalue and eigenvector is

$$e^{(2+4i)t}\begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} = e^{2t}(\cos 4t + i \sin 4t)\begin{bmatrix} 5 \\ 3 - 4i \end{bmatrix} = e^{2t}\begin{bmatrix} 5\cos 4t & + & 5i \sin 4t \\ (3\cos 4t + 4\sin 4t) & + & i(-4\cos 4t + 3\sin 4t) \end{bmatrix}.$$

The real and imaginary parts provide independent real solutions. Thus, the general solution is

$$\vec{x} = e^2 t\left(c_1\begin{bmatrix} 5\cos 4t \\ 3\cos 4t + 4\sin 4t \end{bmatrix} + c_2\begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 3\sin 4t \end{bmatrix}\right). \ \square$$

3. Find a series solution about $x_0 = 0$ for

$$(x-2)y'' + 2y' - xy = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - 2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1}.$$

In the first and third sums, let $n - 1 = k - 2$, so $n = k - 1$. When $n = 0$, $k = 1$, so the series become

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} = \sum_{k=1}^{\infty} (k-1)(k-2)a_{k-1} x^{k-2} = \sum_{n=1}^{\infty} (n-1)(n-2)a_{n-1} x^{n-2},$$

$$2\sum_{n=0}^{\infty} n a_n x^{n-1} = 2\sum_{k=1}^{\infty} (k-1)a_{k-1} x^{k-2} = 2\sum_{n=1}^{\infty} (n-1)a_{n-1} x^{n-2}.$$

In each case, I obtained the second equality by replacing the dummy variable k with n.
In the fourth sum, let $n + 1 = k - 2$, so $n = k - 3$. When $n = 0$, $k = 3$, so the series become

$$-\sum_{n=0}^{\infty} a_n x^{n+1} = -\sum_{k=3}^{\infty} a_{k-3} x^{k-2} = -\sum_{n=3}^{\infty} a_{n-3} x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
Now the differential equation is

$$\sum_{n=1}^{\infty} (n-1)(n-2)a_{n-1} x^{n-2} - 2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} (n-1)a_{n-1} x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0.$$

The sum of the coefficients for each power of x on the left must be 0. For $n = 0$ and $n = 1$, I get $0 = 0$. For $n = 2$, I get

$$-4a_2 + 2a_1 = 0, \quad \text{or} \quad a_2 = \frac{1}{2}a_1.$$

(Note that I can't write down the recursion formula beginning at $n = 2$, because the fourth sum hasn't kicked in yet.)

For $n \geq 3$, I have

$$(n-1)(n-2)a_{n-1} - 2n(n-1)a_n + 2(n-1)a_{n-1} - a_{n-3} = 0, \quad \text{so} \quad a_n = \frac{n(n-1)a_{n-1} - a_{n-3}}{2n(n-1)}.$$

The series solution is

$$y = a_0\left(1 - \frac{1}{12}x^3 - \frac{1}{24}x^4 - \frac{1}{48}x^5 - \frac{13}{1440}x^6 - \cdots\right) + a_1\left(x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{12}x^4 + \frac{7}{240}x^5 + \cdots\right). \ \square$$

4. Find $y''(1)$, $y'''(1)$, and $y^{(4)}(1)$ if

$$xy'' + (x^2 + 1)y' - 3y = 0, \qquad y(1) = 1, y'(1) = 0.$$

Use them to write down the terms of degree ≤ 4 for the series solution about $x_0 = 1$.

Plug in $x = 1$:

$$(1)y''(1) + (2)y'(1) - 3y(1) = 0, \quad y''(1) = 3.$$

Differentiate the equation:

$$xy''' + y'' + 2xy' + (x^2 + 1)y'' - 3y' = 0.$$

Plug in $x = 1$:

$$(1)y'''(1) + y''(1) + (2)y'(1) + (2)y''(1) - 3y'(1) = 0, \quad y'''(1) = -9.$$

Differentiate the previous equation:

$$xy^{(4)} + y''' + y''' + 2y' + 2xy'' + 2xy'' + (x^2 + 1)y''' - 3y'' = 0.$$

Plug in $x = 1$:

$$(1)y^{(4)}(1) + y'''(1) + y'''(1) + 2y'(1) + 2y''(1) + 2y''(1) + (2)y'''(1) - 3y'(1) = 0, \quad y^{(4)}(1) = 33.$$

The series is

$$y = 1 + 0 \cdot x + \frac{3}{2!}x^2 + \frac{(-9)}{3!}x^3 + \frac{33}{4!}x^4 + \cdots. \square$$