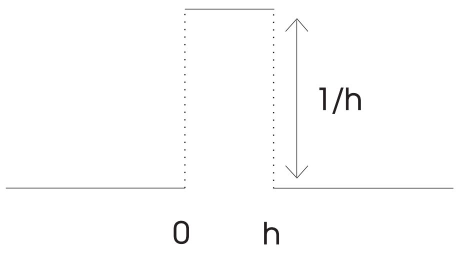
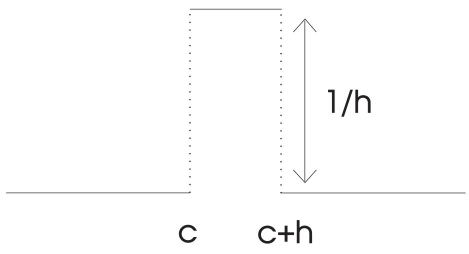
3-3-1994

Impulse functions.

Let $f_h(t)$ be the function which is $\dfrac{1}{h}$ from 0 to h and 0 elsewhere. Then $f_h(t-c)$ is the same function translated to start at c.



$f_h(t)$



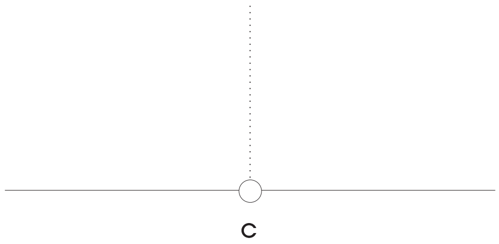
$f_h(t-c)$

Note that

$$\int_c^{c+h} f_h(t-c)\,dt = \int_c^{c+h} \frac{1}{h}\,dt = 1,$$

independent of c and h. That is, the area under the bump is 1, and this obviously doesn't change if I translate it to the left or right.

As $h \to 0+$, $\dfrac{1}{h} \to +\infty$, and the graph becomes an infinite "spike":



Let

$$\delta(t-c) = \lim_{h\to 0+} f_h(t-c).$$

Note that $\delta(t-c)$ isn't a "real" function. It is called the **Dirac delta function**, and is an example of a **generalized function**. From a physical point of view, it represents a sharp impulse applied to a system (hence the name "impulse function").

$\delta(t-c)$ is actually shorthand for a limiting process; when you do computations with it, it will occur inside integrals. So I'll now show how to integrate $\delta(t-c)$ against an arbitrary function:

$$\int_0^\infty \delta(t-c)g(t)\,dt = \int_0^\infty \lim_{h\to 0} f_h(t-c)g(t)\,dt = \lim_{h\to 0}\int_0^\infty f_h(t-c)g(t)\,dt = \lim_{h\to 0}\int_c^{c+h}\frac{1}{h}g(t)\,dt.$$

The $\lim_{h\to 0}\dfrac{1}{h}$ looks like a **difference quotient**, which suggests the next step in the computation:

$$\lim_{h\to 0}\int_c^{c+h}\frac{1}{h}g(t)\,dt = \lim_{h\to 0}\frac{\displaystyle\int_0^{c+h} g(t)\,dt - \int_0^c g(t)\,dt}{h} = \left(\frac{d}{dt}\int_0^t g(u)\,du\right)\bigg|_{t=c} = g(c).$$

The first equality comes from the fact that

$$\int_c^{c+h} g(t)\,dt = \int_0^{c+h} g(t)\,dt - \int_0^c g(t)\,dt.$$

Therefore,

$$\int_0^\infty \delta(t-c)g(t)\,dt = g(c).$$

Example. If $g(t) = 1$,

$$\int_0^\infty \delta(t-c)g(t)\,dt = g(c) = 1.\ \square$$

Example. If $g(t) = e^{-st}$,

$$\mathcal{L}\left(\delta(t-c)\right) = \int_0^\infty \delta(t-c)e^{-st}\,dt = e^{-sc}.\ \square$$

Example. You can use the formula above to compute the transform of $\delta(t-c)h(t)$ for some h. For example,

$$\mathcal{L}\left(\cos t\cdot\delta(t-c)\right) = \int_0^\infty \delta(t-c)\cos t\, e^{-st}\,dt = (\cos c)e^{-sc}.\ \square$$

Example. A mass on a spring experiences an acceleration proportional to its displacement from its equilibrium point. The mass is released 1 meter below its equilibrium point. After π seconds, the mass is hit by a Borg tractor beam, which gives the mass a sudden shove. The equation of motion is

$$\frac{d^2x}{dt^2} + 9x = 3\delta(t-\pi), \qquad x(0) = 1, \quad x'(0) = 0,$$

where x is the displacement from equilibrium. Find the displacement at time t.

Transform both sides:

$$\left(0 - s + s^2X(s)\right) + 9X(s) = 3e^{-\pi s}.$$

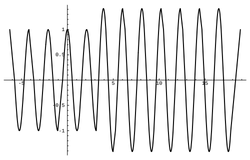
Solve for $X(s)$:

$$X(s) = \frac{s}{s^2+9} + \frac{3}{s^2+9}e^{-\pi s}.$$

Invert the transform to solve for x. (The $e^{-\pi s}$ means I need a a step function shift on the second term.)

$$x(t) = \cos 3t + \sin 3(t-\pi)u_\pi(t).$$

Here is a graph of the solution:



The mass begins oscillating at its natural frequency. The impact of the tractor beam simply increases the amplitude of the oscillations. $\square$