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Linear independence. The Wronskian

A set of vectors $\{v_1, \ldots, v_n\}$ is **independent** if

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad \text{implies} \quad a_1 = \cdots = a_n = 0.$$

.

A set which is not independent is **dependent**.

Example. Determine whether

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is independent.

Suppose

$$a \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

I need to show that $a = b = c = 0$. The equivalent matrix equation is

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set up the coefficient matrix and row reduce. The last column consists of only zeros, and this will never change no matter what row operations are applied. Hence, I'll omit the last column and put it back at the end.

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Digression: Here's how to do the row reduction using *Mathematica*. First, enter the matrix:

```
mat = {{1, 3, 1}, {1, 0, -1}, {0, 2, 1}}
```

A matrix is entered as a list of its rows. Each row is a list of numbers in the row. A list is constructed using curly bracket (`{` and `}`).

Now row reduce the matrix:

```
RowReduce[mat]
```

If you want to see the result displayed in standard matrix format, do

```
RowReduce[mat] // MatrixForm
```

Finally, this problem doesn't require row reduction. You could just use the `Solve` command to solve the original system of equations:

```
Solve[{a + 3b + c == 0, a - c == 0, 2b + c == 0}, {a, b, c}]
```

I'll continue with the example. Remembering that there is an additional all-zero column which I'm not writing, this is the same as the system

$$\begin{aligned} a \qquad\qquad &= 0 \\ b \qquad &= 0 \\ c &= 0 \end{aligned}$$

Therefore, the set of vectors is independent. $\square$

Example. Determine whether

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} \right\}$$

is independent.

Following the set up of the previous example, I set up the matrix equation

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Construct the matrix of coefficients and row reduce. Again, I'm suppressing the all-zero column on the right.

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is equivalent to the system

$$\begin{array}{ccccccc} a & & & + & 2c & = & 0 \\ & & b & - & c & = & 0 \end{array}$$

This implies $a = -2c$ and $b = c$. Set $c = 1$. Then $a = -2$ and $b = 1$. Thus,

$$(-2) \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

I've found a nontrivial linear relationship among the vectors. Therefore, the set of vectors is dependent. $\square$

Example. Any set of vectors containing the zero vector $\vec{0}$ is dependent. For I can write (say)

$$42 \cdot \vec{0} = \vec{0},$$

and that is a nontrivial $(42 \neq 0)$ linear combination of vectors in the set adding up to $\vec{0}$. $\square$

Example. Independence means, roughly, that there is no *redundancy* — you can't build a vector in the set out of the others. It is reasonable, then, that *in* $\mathbf{R}^n$ *a set containing more than n vectors must be dependent.*

For example, the set

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1492 \end{bmatrix}, \begin{bmatrix} 5 \\ 1776 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ \pi \\ 2001 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ -11 \\ 4 \end{bmatrix} \right\}$$

must be dependent. You know this *without* solving a system of equations because we have four 3-dimensional vectors.

Warning: It's not true that a set with fewer than n vectors in $\mathbf{R}^n$ is necessarily independent. (Example: Take two vectors in $\mathbf{R}^3$ which are multiples of one another.) $\square$

Example. It is easy to check independence for 2-dimensional vectors. For example,

$$\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \end{bmatrix}\right\} \quad \text{is independent}$$

$$\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}\right\} \quad \text{is independent}$$

In fact, two 2-dimensional vectors are independent if and only if they aren't multiples.
Notice that

$$\begin{vmatrix} 1 & 2 \\ -3 & -7 \end{vmatrix} = -1 \neq 0, \quad \text{while} \quad \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} = 0.$$

In other words, suppose you have two 2-dimensional vectors. Construct a (square) matrix using the vectors as its columns. The vectors are independent if and only if the determinant of the matrix is nonzero.

This works in general, and is motivation for the discussion of the bf Wronskian which follows. □

If f and g are differentiable functions, the **Wronskian** of f and g is

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

Example. The Wronskian of x^2 and x is

$$W(x^2, x) = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2. \;\square$$

Example. The differential equation $(D - 3)(D + 5)y = 0$ has solutions e^{3x} and e^{-5x}. The Wronskian is

$$W(e^{3x}, e^{-5x}) = \begin{vmatrix} e^{3x} & e^{-5x} \\ 3e^{3x} & -5e^{-5x} \end{vmatrix} = -8e^{-2x}. \;\square$$

If f and g are differentiable functions, f and g are **independent** if

$$c_1 f(x) + c_2 g(x) = 0 \quad \text{implies} \quad c_1 = c_2 = 0.$$

You can use the Wronskian to tell when two functions are independent. Suppose f and g are differentiable on an open interval (a, b).

1. If $W(f, g) \neq 0$ at any point in (a, b), then f and g are independent on (a, b).
2. If f and g are dependent on (a, b), then $W(f, g) \equiv 0$ on (a, b).

Example. Consider the functions e^{3x}, e^{-5x}. From the previous example,

$$W(e^{3x}, e^{-5x}) = -8e^{-2x}.$$

$-8e^{-2x}$ is *never* 0. e^{3x} and e^{-5x} are independent on $(-\infty, +\infty)$. □

To understand why the test works, suppose that $W(f,g)(z) \neq 0$, where $a < z < b$. If

$$c_1 f(x) + c_2 g(x) = 0$$

is an *identity*, then I can differentiate it:

$$c_1 f'(x) + c_2 g'(x) = 0.$$

Write the equations in matrix form:

$$\begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Plug in z:

$$\begin{bmatrix} f & g \\ f' & g' \end{bmatrix} (z) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant of the square matrix is $W(f,g)(z)$. Since it is nonzero, I can invert the matrix:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f & g \\ f' & g' \end{bmatrix}^{-1} (z) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is, $c_1 = c_2 = 0$, so f and g are independent.

Now consider a linear differential equation

$$y'' + p(x)y' + q(x)y = 0.$$

Two solutions y_1 and y_2 are **independent** if

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{implies} \quad c_1 = c_2 = 0.$$

The **Wronskian** of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

When you write down the "general solution" to the equation, you must ensure that we don't write down a solution with "redundancies". That is, you must ensure that the functions that make up the general solution are *independent*.

Example. The equation $(D-1)^2 y = 0$ has characteristic equation $(m-1)^2 = 0$. The root is $m = 1$ (double).

You know that you can't write $y = c_1 e^x + c_2 e^x$, because e^x and e^x aren't independent:

$$W(e^x, e^x) = \begin{vmatrix} e^x & e^x \\ e^x & e^x \end{vmatrix} = 0.$$

e^x and xe^x are independent solutions:

$$W(e^x, e^x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

e^{2x} never vanishes, so e^x and xe^x are independent. □

If y_1, y_2 are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on some interval on which p and q are continuous, the phenomenon observed in the last example occurs. Exactly one of the following is true:

- The solutions y_1 and y_2 are dependent, and the Wronskian $W(y_1, y_2)$ is 0 on the whole interval.
- The solutions y_1 and y_2 are independent, and the Wronskian $W(y_1, y_2)$ is never 0 on the interval.

It's also important to know when solutions y_1 and y_2 give the general solution to the equation $y'' + p(x)y' + q(x)y = 0$. This means that given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$, there should be a solution $y = c_1y_1 + c_2y_2$ satisfying these initial conditions, for some choice of c_1 and c_2. If this is true, then $\{y_1, y_2\}$ is a **fundamental set** of solutions.

In fact, the Wronskian provides a way of telling whether $\{y_1, y_2\}$ is a **fundamental set** of solutions.

- If $W(y_1, y_2)(x_0) \neq 0$, then there are values of c_1 and c_2 for which $y = c_1y_1 + c_2y_2$ is a solution satisfying the initial conditions.

(Note that the results above imply that the solutions are independent.)

There are actually two assertions here. First, I should check that $c_1y_1 + c_2y_2$ *is* a solution; that is, it satisfies

$$y'' + p(x)y' + q(x)y = 0.$$

This is a routine computation, based on the fact that the equation is *linear*.

Second, I need to show that c_1 and c_2 exist. I must have

$$c_1y_1(x_0) + c_2y_2(x_0) = y_0, \quad c_1y_1'(x_0) + c_2y_2'(x_0) = y_0'.$$

Write the equation in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}(x_0)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

As in an earlier argument, the fact that the Wronskian is nonzero at x_0 allows me to invert the square matrix and thereby solve for c_1 and c_2.

Finally, it is possible to compute the Wronskian of a pair of independent solutions to an equation

$$y'' + p(x)y' + q(x)y = 0$$

without knowing the solutions. Well, that's a little optimistic — the formula I'll derive gives the Wronskian up to a constant.

Suppose y_1, y_2 are independent solutions to the equation above. This means that

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \quad y_2'' + p(x)y_2' + q(x)y_2 = 0.$$

Multiply the first equation by $-y_2$ and the second by y_1, then add:

$$\begin{array}{ccccccc} -y_2y_1'' & - & p(x)y_2y_1' & - & q(x)y_2y_1 & = & 0 \\ y_1y_2'' & + & p(x)y_1y_2' & + & q(x)y_1y_2 & = & 0 \\ \hline (y_1y_2'' - y_2y_1'') & + & p(x)(y_1y_2' - y_1'y_2) & & & = & 0 \end{array}$$

Now the Wronskian is $W = y_1y_2' - y_1'y_2$, so its derivative is

$$W' = y_1'y_2' + y_1'y_2'' - y_1''y_2 - y_1'y_2' = y_1'y_2'' - y_1''y_2.$$

Therefore, the equation above can be written as

$$W' + p(x)W = 0, \quad \text{or} \quad \frac{dW}{dx} = -p(x)W.$$

This equation can be solved by separating variables:

$$\int \frac{dW}{W} = -\int p(x)\,dx,$$

$$\ln|W| = -\int p(x)\,dx + c,$$

$$W = c_0 \exp\left(-\int p(x)\,dx\right).$$

Observe that the exponential term cannot be 0. Thus, W is either identically 0 (if $c_0 = 0$) or never 0 (if $c_0 \neq 0$), exactly as I observed earlier.

Example. $y'' - 4xy' + (\sin x)y = 0$.

The Wronskian is

$$W = c_0 \exp \int -(-4x)\,dx = c_0 e^{2x^2}.$$

Notice that the Wronskian has been determined up to a constant, even though the homogeneous solutions aren't known. □

