4-30-1994

Popular mistakes.

1. $D^2(D+3)^3 y = 0$

The solution is

$$y = c_1 + c_2 x + c_3 e^{-3x} + c_4 x e^{-3x}.$$

The D^2 term is *not* a quadratic; it is $(D-0)^2$. $\square$

---

2. (Bogus integration) Here's a popular one:

$$\int \frac{dx}{x+\sqrt{x}} \stackrel{?}{=} \ln\left|x+\sqrt{x}\right| + C$$

This doesn't work, as you can check by differentiating. You could do this integral correctly using the exploding power trick: $x = u^2$.

You can do this integral by parts:

$$\int x^3 e^x\, dx$$

But this isn't the way to do it:

$$\begin{array}{cccc}
 & \dfrac{d}{dx} & & \int dx \\
+ & e^x & & x^3 \\
 & & \searrow & \\
- & e^x & & \dfrac{1}{4}x^4 \\
 & & \searrow & \\
+ & e^x & & \dfrac{1}{20}x^5 \\
 & & \searrow & \\
- & e^x & & \dfrac{1}{120}x^6 \\
\vdots & \vdots & & \vdots
\end{array}$$

The powers are getting larger, so the integral is getting more complicated. The right way is to put the powers of x into the $\dfrac{d}{dx}$ column, where they will decrease.

You also ought to know that certain integrals are undoable. If you come up with one in a problem, it is quite likely that you've made a mistake. Plowing ahead in such a circumstance is a bad idea. For example, these are undoable, in the sense that you can't find an antiderivative in terms of elementary functions:

$$\int e^{x^2}\, dx, \quad \int \frac{\sin x}{x}\, dx, \quad \int \frac{e^x}{x}\, dx, \quad \int \sin(x^2)\, dx. \ \square$$

---

3. $y' - \dfrac{4}{x} y = \dfrac{x^5}{x+1}$

This is indeed first-order linear, but $P = -\dfrac{4}{x}$, not $\dfrac{4}{x}$. Thus, the integrating factor is

$$I = \exp \int -\frac{4}{x}\, dx = x^{-4}.$$

Then

$$yx^{-4} = \int \frac{x}{x+1}\,dx.$$

Note that

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}.$$

So

$$yx^{-4} = x - \ln|x+1| + C.$$

A similar mistake occurs in testing for exactness. The equation

$$(\sin x - 3x^2 y)\,dx - (x^3 + \cos y)\,dy = 0$$

is exact:

$$M_y = -3x^2 \quad \text{and} \quad N_x = -3x^2.$$

The minus sign belongs to N: $N = -(x^3 + \cos y$. □

---

5. The transform of a product *is not* the product of the transforms:

$$\mathcal{L}^{-1}\left(\frac{1}{s^4 + 10s^2 + 9}\right) \stackrel{?}{=} \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right).$$

Instead, use the convolution formula:

$$\mathcal{L}^{-1}\left(\frac{1}{s^4 + 10s^2 + 9}\right) = \sin t \star \frac{1}{3}\sin 3t = \frac{1}{3}\int_0^t \sin(t-u)\sin 3u\,du.$$

You can compute the integral using repeated parts, or the angle addition formula for sine. □

---

6. You can't apply partial fractions to

$$\frac{6e^{-2s}}{s(s-2)(s+1)}.$$

Partial fractions applies to $\dfrac{\text{polynomial}}{\text{polynomial}}$; if you want to use it here, use it on $\dfrac{1}{s(s-2)(s+1)}$. Then apply the step function shift. □

---

7. $\mathcal{L}^{-1}\left(\dfrac{s}{(s-1)^2+16}e^{-3s}\right) \stackrel{?}{=} e^t \cos 4t u_3(t)$

Look at the formula you're trying to apply:

$$\mathcal{L}^{-1}\left(F(s-c)\right) = e^{ct} f(t).$$

Then

$$F(s-c) = \frac{s}{(s-1)^2+16}e^{-3s}.$$

What is F? What is c? If you say $c = 1$, then to match the left side everything should be in terms of $s-1$'s . . .

In fact, the e^{-3s} implies a step function shift:

$$\mathcal{L}^{-1}\left(e^{-sc}F(s)\right) = u_c(t)f(t-c).$$

Match up:

$$e^{-sc}F(s) = \frac{s}{(s-1)^2+16}e^{-3s}.$$

Clearly, $s = 3$, and

$$F(s) = \frac{s}{(s-1)^2+16}.$$

Break F into two pieces:

$$F(s) = \frac{s-1}{(s-1)^2+16} + \frac{1}{(s-1)^2+16}.$$

Hence,

$$f(t) = e^t \cos 4t + \frac{1}{4}e^t \sin 4t.$$

Since $c = 3$,

$$f(t-c) = f(t-3) = e^{t-3}\cos 4(t-3) + \frac{1}{4}e^{t-3}\sin 4(t-3).$$

So

$$\mathcal{L}^{-1}\left(\frac{s}{(s-1)^2+16}e^{-3s}\right) = u_3(t)\left(e^{t-3}\cos 4(t-3) + \frac{1}{4}e^{t-3}\sin 4(t-3)\right). \square$$

8.

$$A = \begin{bmatrix} -3 & 0 & -2 \\ 6 & 1 & 3 \\ 10 & 0 & 6 \end{bmatrix}.$$

What are the eigenvectors for the eigenvalue $\lambda = 1$?

$$A - I = \begin{bmatrix} -4 & 0 & -2 \\ 6 & 0 & 3 \\ 10 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $a + \frac{1}{2}c = 0$, so $a = -\frac{1}{2}c$ (not $a = \frac{1}{2}c$, and not $a = \frac{1}{2}b$). So

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}c \\ b \\ c \end{bmatrix} = b \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}. \square$$

---

9. Solve

$$\vec{x}\,' = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}\vec{x}.$$

You must write your answer in terms of real numbers and functions.
It turns out that $\langle 1+i, 2\rangle$ is an eigenvector for the eigenvalue $\lambda = -2+i$.
Don't leave the answer like this:

$$\vec{x} = c_1 e^{(-2+i)t}\begin{bmatrix} 1+i \\ 2 \end{bmatrix} + c_2 e^{(-2-i)t}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}.$$

Instead, expand $e^{(-2+i)t}\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ and take real and imaginary parts to get two independent solutions:

$$\vec{x} = c_1 e^{-2t}\begin{bmatrix} \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} cost + \sin t \\ 2\sin t \end{bmatrix}. \square$$

---

11. $y'' + y = \tan x$

You can't apply undetermined coefficients here, because $\tan x$ is not a solution of a linear constant coefficients homogeneous equation (i.e. $\tan x$ is not killed by a "D-thingy").

Use variation of parameters instead.

However, if you use variation of parameters, remember to put the equation into standard form. E.g. x and $\dfrac{1}{x}$ are independent homogeneous solutions of

$$x^2 y'' + xy' - y = x\ln x.$$

However, you must divide through by x^2 before using variation of parameters:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{\ln x}{x}. \square$$