5-3-1994

Liapunov Stability.

A **Liapunov function** provides a way of determining whether a critical point is stable, asymptotically stable, or unstable. I'm going to begin by stating the main result, so you can see where this going — after that, I will define the terms.

Assume:

a. $\vec{x}\,' = \vec{f}(\vec{x})$ is an autonomous system. (Most of the examples will be 2-dimensional systems.)

b. $\vec{0}$ is an isolated critical point. (**Isolated** means that you can find a circle centered at the origin in which it's the only critical point.)

c. There is a function $H(x, y)$ (in the 2-dimensional case) which is continuous at $\vec{0}$ and has continuous first partials, such that H is **positive definite** at $\vec{0}$.

Then:

1. If $-\nabla H \cdot \vec{x}\,'$ is positive semidefinite, then $\vec{0}$ is stable.
2. If $-\nabla H \cdot \vec{x}\,'$ is positive definite, then $\vec{0}$ is asymptotically stable.
3. If $\nabla H \cdot \vec{x}\,'$ is positive definite, then $\vec{0}$ is unstable.

Note: You may prefer the following version:

1'. If $\nabla H \cdot \vec{x}\,'$ is negative semidefinite, then $\vec{0}$ is stable.

2'. If $\nabla H \cdot \vec{x}\,'$ is negative definite, then $\vec{0}$ is asymptotically stable.

3'. If $\nabla H \cdot \vec{x}\,'$ is positive definite, then $\vec{0}$ is unstable.

In this version, you are always computing the same thing — $\nabla H \cdot \vec{x}\,'$ — and the conclusion changes according to whether it is negative or positive (semi)definite.

The first thing I need to do is define the terms used in the statement of the result.

A function $H(x, y)$ is **positive definite** if $H(0, 0) = 0$ and $H(x, y) > 0$ for points (x, y) close to, but not including, $(0, 0)$. ("Close to $(0, 0)$" refers to all points inside a sufficiently small circle centered at the origin.)

A function $H(x, y)$ is **positive semidefinite** if $H(0, 0) = 0$ and $H(x, y) \geq 0$ for points (x, y) close to, but not including, $(0, 0)$.

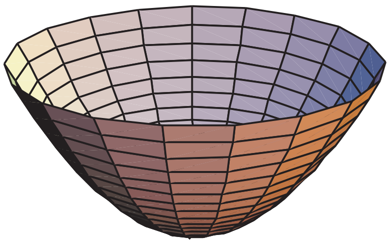
(In other words, **definite** is used for a strict inequality, while **semidefinite** includes the possibility of equality.)

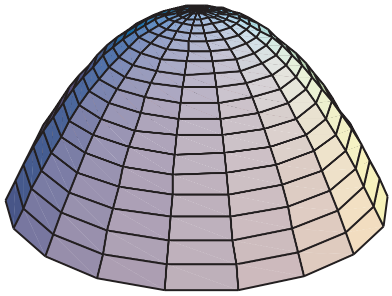
For **negative definite** and **negative semidefinite**, just reverse the inequalities.

A function $H(x, y)$ is **negative definite** if $H(0, 0) = 0$ and $H(x, y) < 0$ for points (x, y) close to, but not including, $(0, 0)$. ("Close to $(0, 0)$" refers to all points inside a sufficiently small circle centered at the origin.)

A function $H(x, y)$ is **negative semidefinite** if $H(0, 0) = 0$ and $H(x, y) \leq 0$ for points (x, y) close to, but not including, $(0, 0)$.

Pictorially, if H is positive definite, the origin looks like a (strict) local min; if H is negative definite, the origin looks like a (strict) local max.





Next, I want to explain the $\nabla H \cdot \vec{x}\,'$ term, and give an intuitive justification for the result.

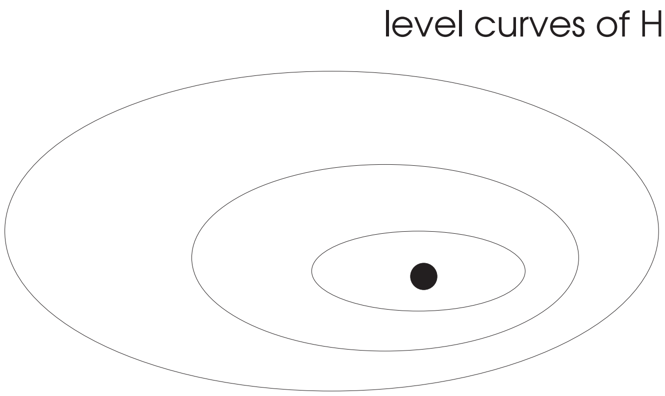
Many examples of Liapunov functions arise as energy functions, where the system to be solved describes a physical situation. The stability theorem I stated says, in this case, that a stable point occurs when the potential energy is a minimum. (Note, however, that Liapunov stability applies in situations where H has nothing to do with energy.)

Now the Chain Rule says

$$\nabla H \cdot \vec{x}\,' = D(H \cdot \vec{x})(t).$$

This is the derivative of H along a solution curve $\vec{x}$. So if $\nabla H \cdot \vec{x}\,' < 0$ in a neighborhood of the origin, the energy H decreases as you move along the curve.

Since H is positive definite, the origin looks like a local min. Since energy decreases along a solution curve, it makes sense that the solution curve should move toward the origin. Moreover, this makes sense geometrically. Near the local min at the origin, the level curves look like ovals encircling the critical point:

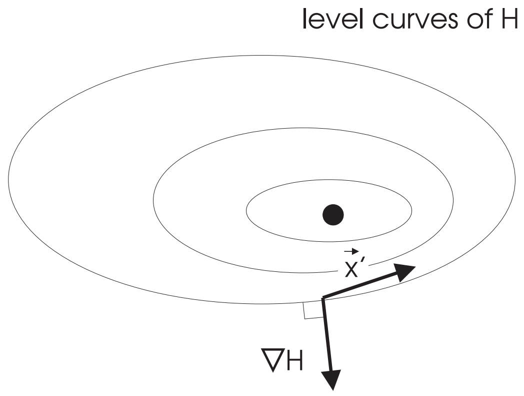


The gradient ∇H is perpendicular to a level curve at a point; since it points in the direction of most rapid increase, it points outward.

If $\nabla H \cdot \vec{x}\,' < 0$, then

$$\cos\theta = \frac{\nabla H \cdot \vec{x}\,'}{|\nabla H||\vec{x}\,'|} < 0.$$

This means that the angle between ∇H and $\vec{x}\,'$ is *obtuse* — greater than a right angle. As you can see from the picture, this should mean that the solution curves should move inward toward the center.



This fits with the conclusion for this case — namely, that the origin is asymptotically stable (attracts nearby solution curves).

Before I give some examples, here's an easy way to tell whether certain functions are positive or negative definite.

Suppose

$$H(x, y) = ax^2 + bxy + cy^2.$$

1. If $4ac - b^2 > 0$ and $a > 0$, then H is positive definite.
2. If $4ac - b^2 > 0$ and $a < 0$, then H is negative definite.

To see why this works, take first and second partials:

$$H_x = 2ax + by, \quad H_y = bx + 2cy,$$

$$H_{xx} = 2a, \quad H_{yy} = 2c, \quad H_{xy} = b.$$

$H_x(0,0) = H_y(0,0) = 0$, so $(0,0)$ is a critical point. Apply the Second Derivative Test:

(x, y)	H_{xx}	H_{yy}	H_{xy}	Δ
$(0,0)$	$2a$	$2c$	b	$4ac - b^2$

$(0,0)$ is a max or a min if $\Delta = 4ac - b^2 > 0$. In this case, it's a max if $2a < 0$ (i.e. if $a < 0$) and a min if $2a > 0$ (i.e. if $a > 0$). Since (as I noted earlier) negative definite means max and positive definite means min, this is exactly the result stated above.

Please note that this only applies to functions of the form $H(x, y) = ax^2 + bxy + cy^2$. In general, how you test for positive definiteness depends on the particular function H.

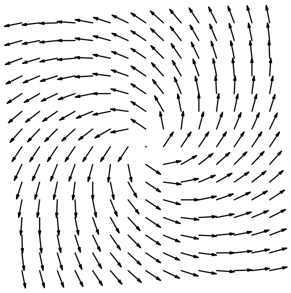
Example.

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = 3x + 2y.$$

In matrix form, this is

$$\vec{x}\,' = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \vec{x}.$$

The origin is a critical point. det $= 13$ and tr $= 4$, so det $> \dfrac{1}{4}\,\mathrm{tr}^2$. The origin is an unstable spiral.



I'll verify the conclusion using the Liapunov function $H(x,y) = x^2 + y^2$.

First, $H(0,0) = 0$ and $H(x,y) > 0$ for (x,y) near the origin (since H is a sum of squares). Thus, H is positive definite.

Next,

$$\nabla H \cdot \vec{x}\,' = \langle 2x, 2y\rangle \cdot \langle 2x - 3y, 3x + 2y\rangle = 4x^2 - 6xy + 6xy + 4y^2 = 4(x^2 + y^2) > 0.$$

Hence, $\nabla H \cdot \vec{x}\,'$ is positive definite. By case 3 of the stability theorem, $\vec{0}$ is unstable. This agrees with the classification based on determinants. $\square$

Example. Consider the system

$$\frac{dx}{dt} = y - \sin x, \quad \frac{dy}{dt} = -4x^3.$$

$(0,0)$ is a critical point; I will classify it using the Liapunov function $H(x,y) = x^4 + \dfrac{1}{2}y^2$. Since $H(0,0) = 0$ and $H(x,y) > 0$ for $(x,y) \neq (0,0)$, H is positive definite.

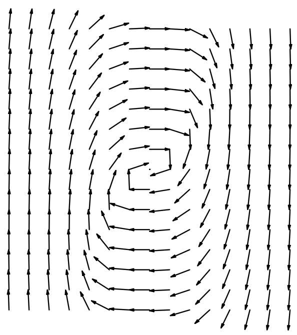
Next,

$$\nabla H \cdot \vec{x}\,' = \langle 4x^3, y\rangle \cdot \langle y - \sin x, -4x^3\rangle = -4x^3 \sin x.$$

Thus, $-\nabla H \cdot \vec{x}\,' = 4x^3 \sin x$. For x near 0, $4x^3 \sin x > 0$:

- If $x > 0$, $x^3 > 0$ and $\sin x > 0$, so $4x^3 \sin x > 0$.
- If $x < 0$, $x^3 < 0$ and $\sin x < 0$, so $4x^3 \sin x > 0$.

Note that $\nabla H \cdot \vec{x}\,' = 0$ on the line $x = 0$. It follows that $-\nabla H \cdot \vec{x}\,'$ is positive semidefinite near $(0,0)$. By case 1, the origin is stable.



$\square$

Example. In general, there is no method for cranking out a Liapunov function in every situation. Finding a Liapunov function requires experience and sometimes, trial and error. In some cases, you can guess the form of a Liapunov function and determine it exactly using something like undetermined coefficients.

Consider the system

$$\frac{dx}{dt} = -x - 4y^2, \quad \frac{dy}{dt} = xy - y^3.$$

The origin is a critical point. I will try a Liapunov function of the form $H(x, y) = x^2 + cy^2$, where $c > 0$ will be determined in the course of the problem.

First, $H(0, 0) = 0$ and $H(x, y) > 0$ for $(x, y) \neq (0, 0)$. Hence, H is positive definite.

Next,

$$\nabla H \cdot \vec{x}\,' = \langle 2x, 2cy\rangle \cdot \langle -x - 4y^2, xy - y^3\rangle = -2x^2 - 8xy^2 + 2cxy^2 - 2cy^4.$$

Notice the positive powers in the first and last terms! If the terms in the middle cancelled, the first and last terms would give $-2x^2 - 2cy^4$, which is always negative. *(To apply the stability theorem, I want $\nabla H \cdot \vec{x}\,'$ to be always positive or always negative.)*

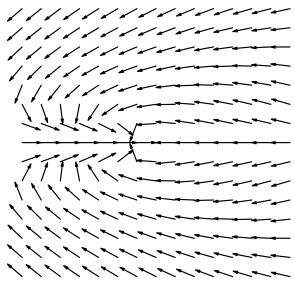
I can make the middle terms cancel by choosing $c = 4$. Doing so, I find that

$$\nabla H \cdot \vec{x}\,' = -2x^2 - 8y^4.$$

Hence, for $(x, y) \neq (0, 0)$,

$$-\nabla H \cdot \vec{x}\,' = 2x^2 + 8y^4 > 0.$$

$-\nabla H \cdot \vec{x}\,'$ is positive definite, so the origin is asymptotically stable.



Observe that the system is almost linear, and the matrix of partial derivatives is

$$A = \begin{bmatrix} 1 & -8y \\ y & x - 3y^2 \end{bmatrix}.$$

At $(0, 0)$,

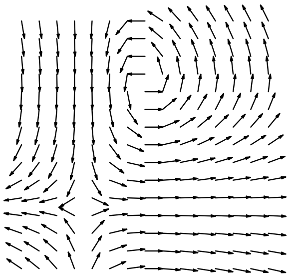
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

So det $= 0$, and the critical point is indeterminate. That is, the method of almost linear systems fails, but I was able to draw a conclusion using Liapunov functions. □

Example. The predator-prey system

$$\frac{dx}{dt} = 2x - xy, \quad \frac{dy}{dt} = xy - y$$

has a critical point at $(1, 2)$. In discussing the predator-prey equations, I asserted that such a critical point would always be stable.



However, the almost linear system method does not produce a conclusion. Here's how to show stability using a Liapunov function.

First, I need to translate everything from the origin to $(1, 2)$. Let

$$u = x - 1, \quad v = y - 2.$$

Now $\dfrac{du}{dt} = \dfrac{dx}{dt}$, $\dfrac{dv}{dt} = \dfrac{dy}{dt}$, so

$$\frac{du}{dt} = 2(u+1) - (u+1)(v+2) = -uv - v,$$

$$\frac{dv}{dt} = (u+1)(v+2) - (v+2) = uv + 2u.$$

I will try a Liapunov function of the form $H(u, v) = f(u) + g(v)$. Then

$$\nabla H \cdot \langle u', v' \rangle = v(-u-1)f'(u) + u(v+2)g'(v).$$

I will try to make this 0. Then the first case will then imply that the critical point is stable.

$$v(-u-1)f'(u) + u(v+2)g'(v) = 0,$$

$$v(u+1)f'(u) = u(v+2)g'(v),$$

$$\frac{u+1}{u}f'(u) = \frac{v+2}{v}g'(v).$$

The left side is a function of u while the right side is a function of v. Hence, both sides must be constant:

$$\frac{u+1}{u}f'(u) = \frac{v+2}{v}g'(v) = k.$$

First,

$$f'(u) = \frac{ku}{u+1} = k - \frac{k}{u+1}, \quad f(u) = ku - k\ln(u+1).$$

Next,

$$g'(v) = \frac{kv}{v+2} = k - \frac{2k}{v+2}, \quad g(v) = kv - 2k\ln(v+2).$$

Take $k = 1$. Then

$$\tilde{H}(u, v) = u - \ln(u+1) + v - \ln(v+2)$$

satisfies $\nabla H \cdot \langle u', v' \rangle = 0$.

I have to adjust $\tilde{H}$ to get a positive definite function H. First, I want $H(0,0)=0$. Now $\tilde{H}(0,0)=-2\ln 2$, so define

$$H(u,v)=\tilde{H}(u,v)-(-2\ln 2)=u-\ln(u+1)+v-\ln(v+2)+2\ln 2.$$

Now $H(0,0)=0$. It only remains to show that H is positive definite. Compute the derivatives:

$$H_u=1-\frac{1}{u+1},\quad H_v=1-\frac{2}{v+2},$$

$$H_{uu}=\frac{1}{(u+1)^2},\quad H_{vv}=\frac{2}{(v+2)^2},\quad H_{uv}=0.$$

Notice that $H_u(0,0)=H_v(0,0)=0$, so $(0,0)$ is a critical point.

(x,y)	H_{uu}	H_{vv}	H_{uv}	Δ
$(0,0)$	1	$\frac{1}{2}$	0	$\frac{1}{2}$

The Second Derivative Test shows that $(0,0)$ is a local min, so H is positive definite. As I noted, $-\nabla H\cdot\langle u',v'\rangle=0$, so $(u,v)=(0,0)$ is stable. Therefore, $(x,y)=(1,2)$ is stable. $\square$