Logistic population growth.

The **logistic equation**

$$\frac{dN}{dt} = aN\left(1 - \frac{N}{K}\right), \quad a > 0,$$

can be solved by separation of variables. The solution is

$$N = \frac{KC_0e^{at}}{1 + C_0e^{at}}.$$

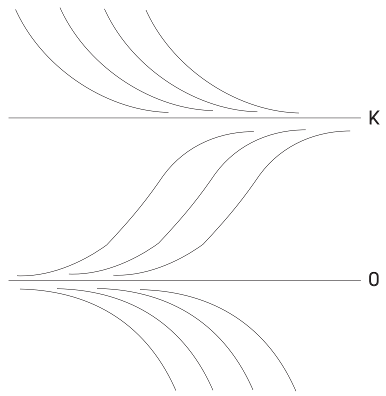
Rewrite the equation

$$N = \frac{K}{C_1e^{-at} + 1}$$

by dividing the top and bottom by C_0e^{at} and renaming the arbitrary constant.

K is called the **carrying capacity**; it represents the largest population that the environment can support.

$\frac{dN}{dt} > 0$ for $0 < N < K$, and $\frac{dN}{dt} < 0$ for $N > K$ and for $N < 0$. Hence, the phase portrait looks like this:



The curves below the line $N = 0$ correspond to negative initial populations, and do not have any physical significance.

Initial populations between 0 and K grow almost exponentially at first. Growth slows as N nears the limiting population K. As $t \to \infty$, $N \to K$.

Initial populations larger than K decrease to K as $t \to \infty$. Physically, the initial population is larger than the environment can support, and hence individuals die off.

The second derivative is

$$\frac{d^2N}{dt^2} = a\frac{dN}{dt} \cdot \left[\left(1 - \frac{N}{K}\right) - N \cdot \frac{1}{K}\right].$$

Observe that $\frac{d^2N}{dt^2} = 0$ for $N = \frac{K}{2}$. Hence, $\frac{dN}{dt}$ has a critical point — in fact, a local max — at $N = \frac{K}{2}$. If you look at the phase portrait, you can see that the population is indeed growing most rapidly when it is half the carrying capacity.

Example. A population of roaches grows logistically in Calvin Butterball's kitchen cabinet, feeding off 65 half-empty cans of Dinty Moore Beef Stew. There are 10 roaches initially, and the carrying capacity of the cabinet is $K = 10000$. The population reaches its maximum growth rate in 4 days. Determine the logistic equation for the growth of the population. Find the number of roaches in the cabinet after 10 days.

Here is the logistic equation:

$$N = \frac{K}{C_1 e^{-at} + 1}.$$

$K = 10000$:

$$N = \frac{10000}{C_1 e^{-at} + 1}.$$

When $t = 0$, $N = 10$:

$$10 = \frac{10000}{C_1 + 1}, \quad C_1 = 999.$$

Now

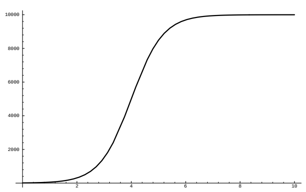
$$N = \frac{10000}{999 e^{-at} + 1}.$$

When $t = 4$, $N = \dfrac{K}{2} = 5000$:

$$5000 = \frac{10000}{999 e^{-4a} + 1}, \quad a = \frac{1}{4} \ln 999.$$

The equation is

$$N = \frac{10000}{999 \exp\left(-\frac{t}{4} \ln 999\right) + 1}.$$



$N(10) \approx 9999.68$, which is 10000 after rounding up. This is the carrying capacity of the cabinet. Calvin is maxed out! □