More on Laplace transforms.

Example. You can use Laplace transforms to solve **linear systems**. For example, consider the system

$$\vec{x}\,' = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Transform the equations:

$$-1 + sX(s) = 2X(s) + 3Y(s),$$

$$1 + sY(s) = X(s) + 2Y(s).$$

From the first equation,

$$Y(s) = -\frac{1}{3} + \frac{1}{3}(s-2)X(s).$$

Plug this into the second equation:

$$1 - \frac{1}{3}s + \frac{1}{3}s(s-2)X(s) = X(s) - \frac{2}{3} + \frac{2}{3}(s-2)X(s).$$

Solve for $X(s)$:

$$X(s) = \frac{s-5}{s^2-4s+1} = \frac{(s-2)-3}{(s-2)^2-3} = \frac{s-2}{(s-2)^2-3} - \frac{3}{(s-2)^2-3}.$$

Therefore,

$$Y(s) = -\frac{1}{3} + \frac{1}{3}\frac{(s-2)^2-3(s-2)}{(s-2)^2-3} = -\frac{s-3}{(s-2)^2-3} = -\frac{s-2}{(s-2)^2-3} + \frac{1}{(s-2)^2-3}.$$

Invert the transforms:

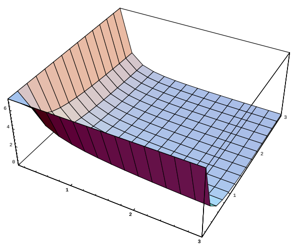
$$x(t) = e^{2t}\cosh\sqrt{3}t - \sqrt{3}e^{2t}\sinh\sqrt{3}t,$$

$$y(t) = -e^{2t}\cosh\sqrt{3}t + \frac{1}{\sqrt{3}}e^{2t}\sinh\sqrt{3}t. \quad \Box$$

Example. The **beta function** is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad x, y > 0.$$

The graph is somewhat boring:



You can use the convolution formula for Laplace transforms to express the beta function in terms of the gamma function. Let

$$f(t) = t^{x-1}, \quad g(t) = t^{y-1}.$$

On the one hand, the convolution is given by

$$f \star g = \int_0^t u^{x-1}(t-u)^{y-1}\, du.$$

Let $u = tw$, so $du = t\, dw$. When $u = 0$, $w = 0$; when $u = t$, $w = 1$. So the integral becomes

$$t^{x+y-1} \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = t^{x+y-1} B(x, y).$$

On the other hand,

$$F(s) = \mathcal{L}[f(t)] = \frac{\Gamma(x)}{s^x}, \quad G(s) = \mathcal{L}[g(t)] = \frac{\Gamma(y)}{s^y}.$$

So by the convolution formula,

$$f \star g = \mathcal{L}^{-1}\left(F(s)G(s)\right) = \mathcal{L}^{-1}\left[\frac{\Gamma(x)\Gamma(y)}{s^{x+}}\right] = \Gamma(x)\Gamma(y)\, \mathcal{L}^{-1}\left(\frac{1}{s^{x+y}}\right) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} t^{x+y-1}.$$

So

$$t^{x+y-1} B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} t^{x+y-1},$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Here is an unusual application of this formula. I already mentioned that $1 \star f(t)$ need not equal $f(t)$ — convolution is not multiplication! Here is another convolution that doesn't work the way you might think.

First, recall that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

You may have seen this in your multivariable calculus course. The idea is that if $I = \int_0^\infty e^{-x^2}\, dx$, then also $I = \int_0^\infty e^{-y^2}\, dy$, so

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

The double integral on the right can be computed by converting to polar, and from there you can solve for I.

Now

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2}\, dt.$$

Let $t = u^2$, so $dt = 2u\, du$. When $u = 0$, $t = 0$, and when $u \to \infty$, $t \to \infty$. So the integral becomes

$$\int_0^\infty e^{-u^2} \frac{1}{u} \cdot 2u\, du = 2 \int_0^\infty e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Now consider the convolution of $\dfrac{1}{\sqrt{t}}$ with itself. It is

$$\frac{1}{\sqrt{t}} \star \frac{1}{\sqrt{t}} = \int_0^t \frac{1}{\sqrt{u}} \frac{1}{\sqrt{t-u}}\, du.$$

Let $u = wt$, so $du = w\,dt$. When $w = 0$, $u = u$, and when $w = 1$, $u = t$. The integral becomes

$$\int_0^1 \frac{t\,dw}{\sqrt{wt}\cdot\sqrt{t-tw}} = \int_0^1 \frac{dw}{\sqrt{w}\cdot\sqrt{1-w}} = \int_0^1 w^{-1/2}(1-w)^{-1/2}\,dw = B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi.$$

That is,

$$\frac{1}{\sqrt{t}} \star \frac{1}{\sqrt{t}} = \pi.$$

The convolution of two nonconstant functions can be a constant! □

Example. Use Laplace transforms to solve

$$y'' - 2y' + y = e^t.$$

Hey! What's going on? Where are the initial conditions $y(0)$, $y'(0)$?
Set $y(0) = c_1$, $y'(0) = c_2$. Transform the differential equation:

$$\left(-c_2 - c_1 s + s^2 Y(s)\right) - 2\left(-c_1 + sY(s)\right) + Y(s) = \frac{1}{s-1}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{(s-1)^3} + \frac{2c_1 + c_2}{(s-1)^2} + \frac{c_1 s}{(s-1)^2}.$$

I know the inverse transforms of the first two terms. For the third,

$$\frac{s}{(s-1)^2} = \frac{s-1+1}{(s-1)^2} = \frac{1}{s-1} + \frac{1}{(s-1)^2}.$$

Therefore,

$$y(t) = \frac{1}{2}t^2 e^t + (2c_1 + c_2)te^t + c_1(e^t + te^t).$$

This *is* the general solution — notice the two arbitrary constants!
Of course, you could solve the original equation using undetermined coefficients or variation of parameters. I think it's roughly the same amount of work. □

Example. Compute

$$\mathcal{L}^{-1}\left(\frac{s}{(s-2)(s^2+1)}e^{-2s}\right).$$

The e^{-2s} means that I will have a step function shift; to apply the step function formula, I need the inverse transform

$$\mathcal{L}^{-1}\left(\frac{s}{(s-2)(s^2+1)}\right).$$

This would probably lead to an obnoxious partial fractions computation; I'll use convolutions instead. First,

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}, \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t.$$

So

$$\mathcal{L}^{-1}\left(\frac{s}{(s-2)(s^2+1)}\right) = e^{2t} \star \cos t = \int_0^t e^{2(t-u)}\cos u\,du = e^{2t}\int_0^t e^{-2u}\cos u\,du.$$

Compute the integral by parts:

$$\begin{array}{ccccc} & \dfrac{d}{du} & & \int du \\ + & e^{-2u} & & \cos u \\ & & \searrow & \\ - & -2e^{-2u} & & \sin u \\ & & \searrow & \\ + & 4e^{-2u} & \rightarrow & -\cos u \end{array}$$

$$\int e^{-2u}\cos u\,du = e^{-2u}\sin u - 2e^{-2u}\cos u - 4\int e^{-2u}\cos u\,du,$$

$$\int e^{-2u}\cos u\,du = \frac{1}{5}\left(e^{-2u}\sin u - 2e^{-2u}\cos u\right) + C.$$

So

$$e^{2t}\int_0^t e^{-2u}\cos u\,du = \frac{e^{2t}}{5}\left[e^{-2u}\sin u - 2e^{-2u}\cos u\right]_0^t = \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{2}{5}e^{2t}.$$

Let

$$g(t) = \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{2}{5}e^{2t}.$$

Then

$$\mathcal{L}^{-1}\left(\frac{s}{(s-2)(s^2+1)}e^{-2s}\right) = g(t-2)u_2(t). \square$$