4-8-1994

Nonhomogeneous linear systems.

You can solve **nonhomogeneous linear systems** of the form

$$\vec{x}\,' = A\vec{x} + \vec{b}$$

using the matrix exponential. (In this equation, A is a matrix of numbers, while $\vec{b}$ is a vector of functions — see the examples below.) The idea is to use an **integrating factor** like the one you used to solve first-order linear equations.

Rewrite the equation in the form

$$\vec{x}\,' - A\vec{x} = \vec{b}.$$

Notice that this looks like the first-order equation $y' + p(x)y = q(x)$, which has the integrating factor

$$I = e^{\int p(x)\,dx}.$$

By analogy, I'll try the integrating factor e^{-At}. Multiply $\vec{x}\,' - A\vec{x} = \vec{b}$ by e^{-At}:

$$e^{-At}\vec{x}\,' - Ae^{-At}\vec{x} = e^{-At}\vec{b}.$$

By the Product Rule,

$$\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}\vec{x}\,' - Ae^{-At}\vec{x}.$$

So

$$\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}\vec{b}.$$

Integrate both sides:

$$e^{-At}\vec{x} = \int e^{-At}\vec{b}\,dt + \vec{c}.$$

Finally,

$$\vec{x} = e^{At}\int e^{-At}\vec{b}\,dt + e^{At}\vec{c}.$$

Notice that the second term $e^{At}\vec{c}$ is the solution to the homogeneous equation $\vec{x}\,' = A\vec{x}$. It plays the role of the *homogeneous* solution, while $e^{At}\int e^{-At}\vec{b}\,dt$ plays the role of the *particular* solution.

One note before I do some examples: You already know how to find e^{At}. You can get e^{-At} from e^{At} by replacing all the t's by $-t$'s.

Example. Solve the nonhomogeneous system

$$\vec{x}\,' = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$; the eigenvalues are $\lambda = 4$ and $\lambda = -2$. I'll compute e^{At} using the iterative algorithm; I'll compute the a_k's first. $a_1 = e^{4t}$, and

$$a_2 = e^{-2t} \star e^{4t} = \int_0^t e^{-2(t-u)}e^{4u}\,du = e^{-2t}\int_0^t e^{6u}\,du = \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t}.$$

Next, I'll compute the B_k's. $B_1 = I$, and

$$B_2 = A - 4I = \begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix}.$$

Therefore,

$$e^{At} = e^{4t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t}\right)\begin{bmatrix} -5 & 5 \\ 1 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} e^{4t} + 5e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & 5e^{4t} + e^{-2t} \end{bmatrix}.$$

Next, I'll compute the particular solution $e^{At}\int e^{-At}\vec{b}\,dt$. I'll do the computation in steps. First,

$$e^{-At} = \frac{1}{6}\begin{bmatrix} e^{-4t} + 5e^{2t} & e^{-4t} - e^{2t} \\ e^{-4t} - e^{2t} & 5e^{-4t} + e^{2t} \end{bmatrix}.$$

Next,

$$e^{-At}\vec{b} = \frac{1}{6}\begin{bmatrix} e^{-4t} + 5e^{2t} & e^{-4t} - e^{2t} \\ e^{-4t} - e^{2t} & 5e^{-4t} + e^{2t} \end{bmatrix}\begin{bmatrix} e^t \\ 2e^t \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3e^{-3t} + 3e^{3t} \\ 11e^{-3t} + e^{3t} \end{bmatrix}.$$

Integrate:

$$\int e^{-At}\vec{b}\,dt = \frac{1}{6}\begin{bmatrix} -e^{-3t} + e^{3t} \\ -\dfrac{11}{3}e^{-3t} + \dfrac{1}{3}e^{3t} \end{bmatrix}.$$

Hence,

$$e^{At}\int e^{-At}\vec{b}\,dt = \frac{1}{36}\begin{bmatrix} e^{4t} + 5e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & 5e^{4t} + e^{-2t} \end{bmatrix}\begin{bmatrix} -e^{-3t} + e^{3t} \\ -\dfrac{11}{3}e^{-3t} + \dfrac{1}{3}e^{3t} \end{bmatrix}.$$

The solution to the original system is

$$\vec{x} = \frac{1}{36}\begin{bmatrix} e^{4t} + 5e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & 5e^{4t} + e^{-2t} \end{bmatrix}\begin{bmatrix} -e^{-3t} + e^{3t} \\ -\dfrac{11}{3}e^{-3t} + \dfrac{1}{3}e^{3t} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} e^{4t} + 5e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & 5e^{4t} + e^{-2t} \end{bmatrix}\vec{c}.\ \square$$

Example. Solve the initial value problem

$$\vec{x}\,' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ 7e^{2t} \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The eigenvalue is $\lambda = 2$ (a triple root). I'll let you check that

$$a_1 = e^{2t}, \quad a_2 = te^{2t}, \quad a_3 = \frac{1}{2}t^2e^{2t},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} e^{2t} & te^{2t} & \dfrac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \quad \text{and} \quad e^{-At} = \begin{bmatrix} e^{-2t} & -te^{-2t} & \dfrac{1}{2}t^2e^{-2t} \\ 0 & e^{-2t} & -te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}.$$

Then

$$e^{-At}\vec{b} = \begin{bmatrix} 1 - 3t + \frac{7}{2}t^2 \\ 3 - 7t \\ 7 \end{bmatrix},$$

$$\int e^{-At}\vec{b}\,dt = \begin{bmatrix} t - \frac{3}{2}t + \frac{7}{6}t^3 \\ 3t - \frac{7}{2}t^2 \\ 7t \end{bmatrix},$$

$$e^{At}\int e^{-At}\vec{b}\,dt = \begin{bmatrix} t + \frac{3}{2}t + \frac{7}{6}t^3 \\ 3t + \frac{7}{2}t^2 \\ 7t \end{bmatrix} e^{2t}.$$

So the solution to the system is

$$\vec{x} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \vec{c} + \begin{bmatrix} t + \frac{3}{2}t + \frac{7}{6}t^3 \\ 3t + \frac{7}{2}t^2 \\ 7t \end{bmatrix} e^{2t}.$$

Plug in the initial condition, remembering that $e^{A\cdot 0} = I$:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{x}(0) = I\cdot\vec{c} + \vec{0}\cdot e^{2t} = \vec{c}.$$

Hence, the solution to the initial value problem is

$$\vec{x} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} t + \frac{3}{2}t + \frac{7}{6}t^3 \\ 3t + \frac{7}{2}t^2 \\ 7t \end{bmatrix} e^{2t}. \quad \square$$