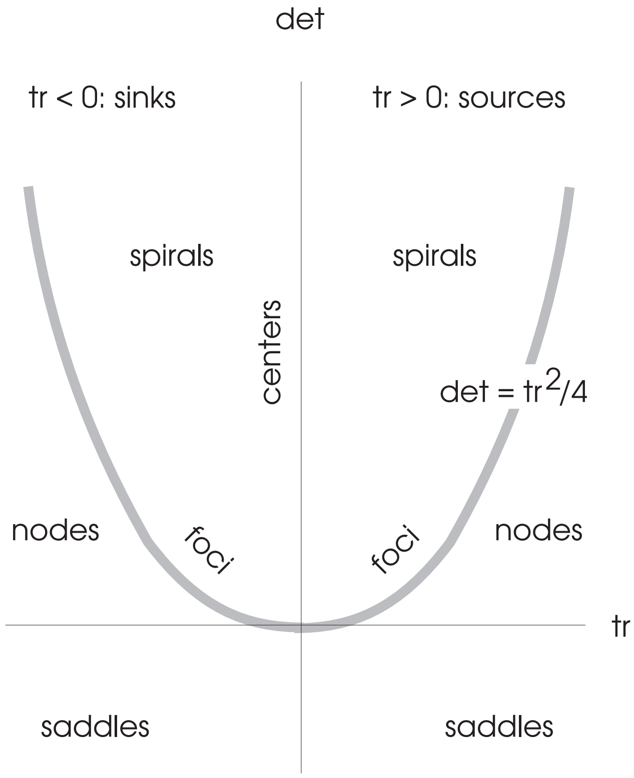
4-16-1994

Phase portraits.

I will discuss the different types of phase portraits that can arise from a 2-dimensional linear system:

$$\vec{x}\,' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}, \quad \text{or} \quad \vec{x}\,' = A\vec{x}.$$

The classification can be summed up by the following diagram:



In this diagram, det is the **determinant** of A and tr is the **trace** of A. You know the determinant is $\det A = ad - bc$; the trace is the sum of the diagonal entries: $\operatorname{tr} A = a + d$. Here is how the determinant and trace are related to the eigenvalues.

The characteristic polynomial is

$$(\lambda - a)(\lambda - d) - bc = (\lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - (\operatorname{tr} A) + (\det A).$$

But if λ_1 and λ_2 are the eigenvalues, they are the roots of the characteristic polynomial. So the characteristic polynomial is also

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Comparing the two expressions for the characteristic polynomial, you can see that

$$\operatorname{tr} A = \lambda_1 + \lambda_2 \quad \text{and} \quad \det A = \lambda_1\lambda_2.$$

In words, the trace is the sum of the eigenvalues, while the determinant is the product of the eigenvalues.

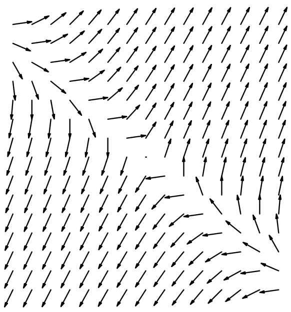
Example. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & \pi \end{bmatrix}$. The trace is $\operatorname{tr} A = 1 + \pi$, while $\det A = \pi - 6$. It follows that the sum of the eigenvalues is $1 + \pi$, while the product of the eigenvalues is $\pi - 6$.

Consider the linear system

$$\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 3 & \pi \end{bmatrix} \vec{x}.$$

$\operatorname{tr} A > 0$, while $\det A < 0$. The phase portrait for this system will be a **saddle**.

Here's the corresponding vector field:



There are two independent eigenvectors. You can see them in the picture as the two lines that pass through the origin. The origin is **unstable**, in the sense that if you follow a solution curve starting at a point near the origin, you will move away from the origin. □

I'll now examine a few of the cases in more detail. First, I want to explain the parabola $\det = \dfrac{1}{4} \operatorname{tr}^2$ that appears in the classification picture.

Recall that the characteristic polynomial is $\lambda^2 - (\operatorname{tr} A)\lambda + (\det A)$.

If $\det = \dfrac{1}{4} \operatorname{tr}^2$, the characteristic polynomial is

$$\lambda^2 - (\operatorname{tr} A)\lambda + \frac{1}{4} \operatorname{tr}^2 = \left(\lambda - \frac{1}{2} \operatorname{tr} A\right)^2.$$

That is, the matrices on the parabola are those with a repeated eigenvalue.

Suppose $\det > \dfrac{1}{4} \operatorname{tr}^2$. These are points *above* the parabola. The characteristic polynomial is

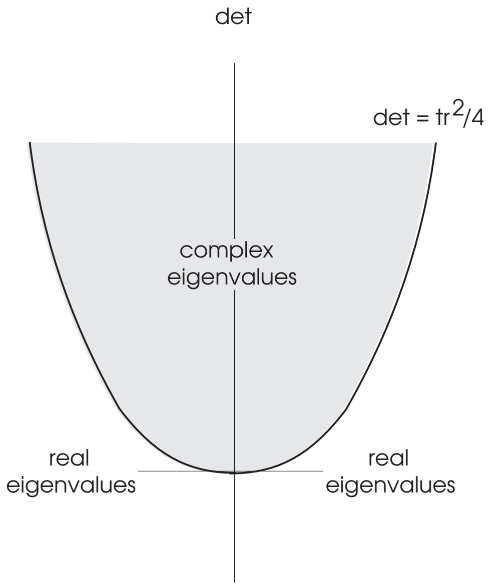
$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - (\operatorname{tr} A)\lambda + \frac{1}{4} \operatorname{tr}^2 + \left(\det A - \frac{1}{4} \operatorname{tr}^2\right) = \left(\lambda - \frac{1}{2} \operatorname{tr} A\right)^2 + \left(\det A - \frac{1}{4} \operatorname{tr}^2\right).$$

If I set this equal to 0, I get

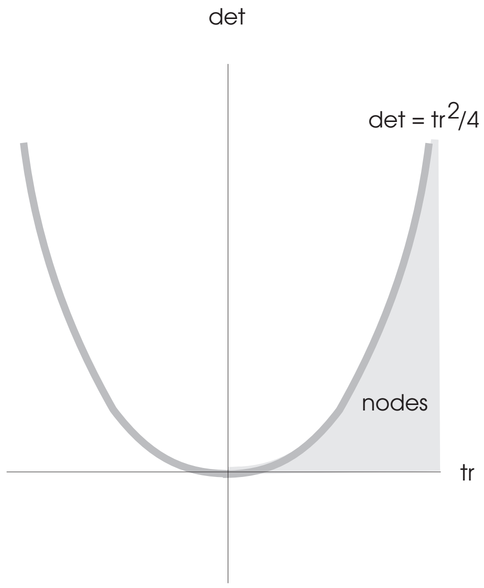
$$\left(\lambda - \frac{1}{2} \operatorname{tr} A\right)^2 = -\left(\det A - \frac{1}{4} \operatorname{tr}^2\right) < 0.$$

It follows that the roots — the eigenvalues — are complex. Notice that these systems are **spirals** or **centers**.

Likewise, $\det < \frac{1}{4}\,\mathrm{tr}^2$ means that the eigenvalues are real.



Now consider the part of the classification diagram where $\mathrm{tr} > 0$, $\det > 0$, and $\det < \frac{1}{4}tr^2$.



Since $\det > 0$ and since det is the product of the eigenvalues, the eigenvalues are both positive or both negative.

Since $\mathrm{tr} > 0$ and since tr is the sum of the eigenvalues, the eigenvalues must be both positive.

Since $\det < \frac{1}{4}tr^2$, the eigenvalues are real and distinct.

Thus, there are two distinct, positive eigenvalues. Say they are λ_1 and λ_2, and $\lambda_1 > \lambda_2 > 0$. If $\vec{v}_1$ is an eigenvector for λ_1 and $\vec{v}_2$ is an eigenvector for λ_2, the solution is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

Factor out $e^{\lambda_1 t}$:

$$\vec{x} = e^{\lambda_1 t}\left(c_1\vec{v}_1 + c_2 e^{(\lambda_2-\lambda_1)t}\vec{v}_2\right).$$

Notice that $\lambda_2 - \lambda_1$ is negative, because λ_1 was the bigger eigenvalue. So as $t \to \infty$, $e^{(\lambda_2-\lambda_1)t} \to 0$, and hence $\vec{x}$ approaches the line of the vector $\vec{v}_1$.

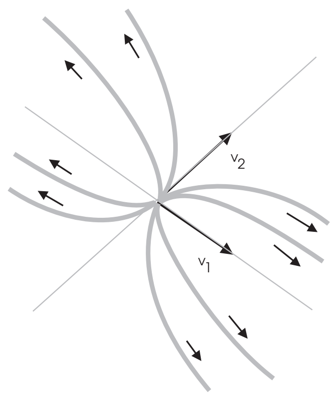
In other words, as $t \to \infty$, the solution curves become parallel to the eigenvector for the bigger eigenvalue.

Now go back to the $\vec{x}$ equation:

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

As $t \to -\infty$, both positive exponentials go to 0, so $\vec{x} \to \vec{0}$. In other words, the solution curves emanate from the origin.

Here's the picture:



I've drawn arrows to show the direction of increasing t. Notice that the curves emanate from the origin, and gradually become parallel to the vector $\vec{v}_1$. This type of phase portrait is called an **(improper) node**.

Now consider the case where $\det > \dfrac{1}{4}\operatorname{tr}^2$. As I observed above, the eigenvalues are complex: suppose they are $\lambda = a \pm bi$.

The following system has eigenvalues $\lambda = a \pm bi$:

$$\vec{x}\,' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \vec{x}.$$

This is not the only possibility, but I will only look at this special case to keep things reasonably simple.

This gives

$$x' = ax + by, \quad y' = -bx + ay.$$

I'm going to convert to polar; I need to do a couple of differentiations first. Using $r^2 = x^2 + y^2$, I get

$$2rr' = 2xx' + 2yy',$$

$$rr' = xx' + yy'. \tag{1}$$

Likewise, $\tan\theta = \dfrac{y}{x}$ gives

$$(\sec\theta)^2\theta' = \frac{1}{x}y' - \frac{y}{x^2}x'. \tag{2}$$

Substitute for x' and y' in equation (1):

$$rr' = x(ax + by) + y(-bx + ay) = a(x^2 + y^2) = ar^2.$$

Cancelling the r's yields $r' = ar$. This is the exponential growth equation, and the solution is

$$r = c_1 e^{at}.$$

Notice that if $a \neq 0$ — that is, if the eigenvalues are not *pure* imaginary numbers — r will increase as $t \to \infty$ or as $t \to -\infty$ (depending on the sign of a).

Now substitute for x' and y' in equation (2):

$$(\sec\theta)^2\theta' = \frac{1}{x}(-bx+ay) - \frac{y}{x^2}(ax+by) = -b\frac{x^2+y^2}{x^2} = -b(\sec\theta)^2.$$

Cancelling the $(\sec\theta)^2$'s yields $\theta' = -b$, so $\theta = -bt + c_2$.

Now $b \neq 0$ — I'm assuming the eigenvalues *are* complex — so by letting $t \to \infty$ or $t \to -\infty$, I can get θ to go to ∞ or $-\infty$ (depending on the sign of b). Since θ is the polar angle, this means the solution curves go round and round the origin.

Combine this with what I know about r. In the case $a \neq 0$, the radial distance r increases or decreases with t. Since the curves also go round and round the origin, this means they are **spirals**.

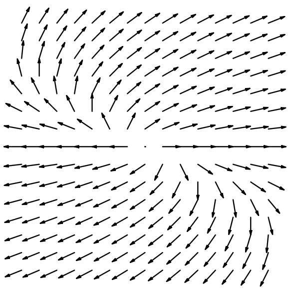
If $a = 0$, then $r = c_1$, a constant. In this case, the solution curves are circles centered at the origin. These are called **centers**.

Remember that this is not the only way a system can have eigenvalues $\lambda = a \pm bi$. In general, centers can look like *ellipses* centered at the origin.

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \quad \det = 4, \quad \operatorname{tr} = 4$$

$\det > 0$, $\operatorname{tr} > 0$, and $\det = \frac{1}{4}\operatorname{tr}^2$. The phase portrait is a **focus** (this is sometimes referred to as a node):

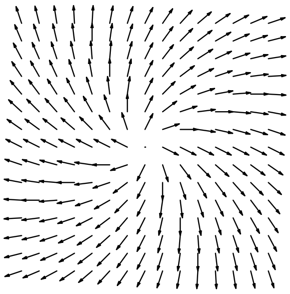


□

Example.

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad \det = 5, \quad \operatorname{tr} = 4$$

$\det > 0$, $\operatorname{tr} > 0$, and $\det > \frac{1}{4}\operatorname{tr}^2$. The phase portrait is a **spiral**:

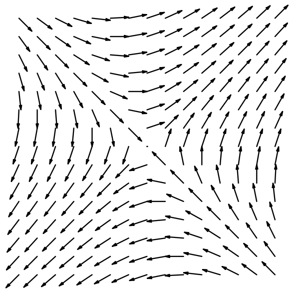


□

Example.

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \det = -8, \quad \mathrm{tr} = 2$$

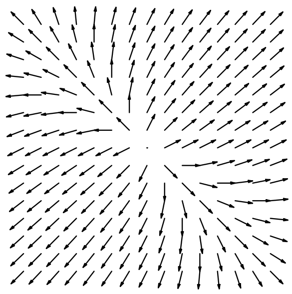
$\det < 0$ and $\mathrm{tr} > 0$. The phase portrait is a **saddle**:



Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \det = 3, \quad \mathrm{tr} = 4$$

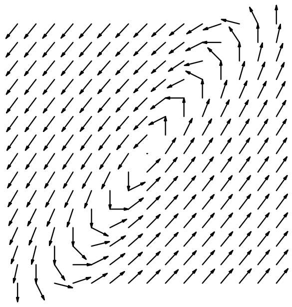
$\det > 0$, $\mathrm{tr} > 0$, and $\det < \dfrac{1}{4}\mathrm{tr}^2$. The phase portrait is an **improper node**:



Example.

$$A = \begin{bmatrix} 2 & -2 \\ 3 & -2 \end{bmatrix}, \quad \det = 2, \quad \mathrm{tr} = 0$$

det > 0 and tr $= 0$. The phase portrait is a **center**:



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