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Nonhomogeneous equations - Reduction of order.

The solution of a nonhomogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = f(x)$$

is related to the solution of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Suppose y_1 is a particular solution to the homogeneous equation. **Reduction of order** "bootstraps" up from this particular solution to the general solution to the original equation.

The idea is to "guess" a general solution of the form $y = vy_1$, where v is to be determined. The derivatives are

$$y' = v'y_1 + vy_1', \qquad y'' = v''y_1 + 2v'y_1' + vy_1''.$$

Therefore,

$$\begin{array}{rclcccl} y'' + p(x)y' + q(x)y & = & v''y_1 & + & 2v'y_1' & + & vy_1'' \\ & & & + & pv'y_1 & + & pvy_1' \\ & & & & & + & qvy_1 \\ & = & v''y_1 & + & (2y_1' + py_1)v' & + & (y_1'' + py_1' + qy_1)v \\ & = & v''y_1 & + & (2y_1' + py_1)v'. & & \end{array}$$

(I used the fact that $y_1'' + py_1' + qy_1 = 0$.)

Thus,

$$v''y_1 + (2y_1' + py_1)v' = f(x).$$

Now set $u = v'$, so $u' = v''$. The equation becomes

$$u'y_1 + (2y_1' + py_1)u = f(x),$$

which is first order linear.

Now all you need to do is to solve the equation for u, find v, and finally obtain the general solution y.

Rather than memorizing the last equation, it is better to understand the method, and to carry out the computation in each case.

Example. First, I'll check that this works in a situation where I know the general solution. Consider the equation

$$y'' - y = 0.$$

The solutions are e^x and e^{-x}. I'll use e^x as y_1, so $y = ve^x$. Then

$$y' = v'e^x + ve^x, \quad y'' = v''e^x + 2v'e^x + ve^x.$$

Substitute:

$$\begin{array}{rclcccl} y'' - y & = & v''e^x & + & 2v'e^x & + & ve^x \\ & & & & & - & ve^x \\ & = & v''e^x & + & 2v'e^x. & & \end{array}$$

Now I have

$$v''e^x + 2v'e^x = 0.$$