Math 228
3-19-1994

Review Session for Test 2.

Example. Find the general solution to

$$x^2y'' + 5xy' + 4y = 0, \qquad x > 0,$$

given that $y_1 = \dfrac{1}{x^2}$ is a solution.

Use reduction of order. Set $y = vy_1$. The derivatives are

$$y' = v'y_1 + vy_1', \qquad y'' = v''y_1 + 2v'y_1' + vy_1''.$$

Plug them into the original equation:

$$\begin{array}{rcccccccc} x^2y'' + 5xy' + 4y & = & x^2v''y_1 & + & 2x^2v'y_1' & + & x^2vy_1'' & & \\ & & & + & 5xv'y_1 & + & 5xvy_1' & & \\ & & & & & + & 4vy_1 & & \\ & = & x^2v''y_1 & + & v'(2x^2y_1' + 5xy_1) & + & v(x^2y_1'' + 5xy_1' + 4y_1) & & \\ & = & x^2v''y_1 & + & v'(2x^2y_1' + 5xy_1) & & & & = 0. \end{array}$$

($x^2y_1'' + 5xy_1' + 4y_1 = 0$ because y_1 is a solution.)
Now substitute $y_1 = \dfrac{1}{x^2}$, $y_1' = -\dfrac{2}{x^3}$:

$$v'' + v'\left(2s^2 \cdot \left(-\frac{2}{x^3}\right) + \frac{5}{x}\right) = 0, \quad v'' + v' \cdot \frac{1}{x} = 0.$$

Let $u = v'$, so $u' = v''$. The equation becomes

$$\frac{du}{dx} + \frac{u}{x} = 0.$$

Separate variables:

$$\int \frac{du}{u} = -\int \frac{dx}{x}, \quad \ln|u| = -\ln|x| + c, \quad u = \frac{c_1}{x}.$$

Put v back:

$$v' = \frac{c_1}{x}, \quad v = c_1 \ln|x| + c_2.$$

Finally,

$$y = vy_1 = c_1 \cdot \frac{\ln|x|}{x^2} + c_2 \cdot \frac{1}{x^2}. \ \square$$

Example. Find the general solution to

$$y'' - 3y' + 2y = \frac{e^x}{1 + e^{-x}}.$$

Use variation of parameters. $y'' - 3y' + 2y = 0$ has characteristic equation $m^2 - 3m + 2 = 0$. The roots are $m = 1$ and $m = 2$, so $y_1 = e^x$ and $y_2 = e^{2x}$ are independent homogeneous solutions.

Set up the variation of parameters equation:

$$\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{e^x}{1+e^{-x}} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{-3x} \begin{bmatrix} 2e^{2x} & -e^{2x} \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{e^x}{1+e^{-x}} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{1+e^{-x}} \\ \dfrac{e^{-x}}{1+e^{-x}} \end{bmatrix}.$$

$$u_1' = \frac{-1}{1+e^{-x}} = \frac{-e^x}{e^x+1}, \quad \text{so} \quad u_1 = -\ln(e^x+1) + c_1.$$

$$u_2' = \frac{e^{-x}}{1+e^{-x}}, \quad \text{so} \quad u_2 = -\ln(1+e^{-x}) + c_2.$$

The solution is

$$y = u_1 y_1 + u_2 y_2 = -e^x \ln(e^x+1) + c_1 e^x - e^{2x} \ln(1+e^{-x}) + c_2 e^{2x}. \ \square$$

Example. Find the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t.$$

Now apply the Convolution formula:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right] = e^{-2t} \star e^{-t}\sin t = \int_0^t e^{-2(t-u)} e^{-u} \sin u \, du = e^{-2t} \int_0^t e^u \sin u \, du.$$

Integrate by parts:

$$\begin{array}{cccc} & \dfrac{d}{du} & & \int du \\ + & e^u & & \sin u \\ & & \searrow & \\ - & e^u & & -\cos u \\ & & \searrow & \\ + & e^u & \rightarrow & -\sin u \end{array}$$

Thus,

$$\int e^u \sin u \, du = -e^u \cos u + e^u \sin u - \int e^u \sin u \, du,$$

$$2 \int e^u \sin u \, du = -e^u \cos u + e^u \sin u,$$

$$\int e^u \sin u \, du = \frac{1}{2}(-e^u \cos u + e^u \sin u) + C.$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)(s^2+2s+2)}\right] = \frac{1}{2}e^{-2t}\left[-e^u\cos u + e^u \sin u\right]_0^t = \frac{1}{2}e^{-2t}\left(-e^t\cos t + e^t\sin t + 1\right). \square$$

Example. Solve for y in terms of t:

$$y'' + 4y = (t-2)u_2(t), \qquad y(0) = 1, \quad y'(0) = 1.$$

Transform both sides:

$$\left(-1 - s + s^2 Y(s)\right) + 4Y(s) = \frac{1}{s^2}e^{-2s}.$$

(What happened on the right? I matched $(t-2)u_2(t)$ with $f(t-c)u_c(t)$. $c = 2$, so $f(t-2) = t-2$. This means $f(t) = t$, so $F(s) = \dfrac{1}{s^2}$.)

Solve for $Y(s)$:

$$Y(s) = \frac{s+1}{s^2+4} + \frac{1}{s^2(s^2+4)}e^{-2s}.$$

First,

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \cos 2t + \frac{1}{2}\sin 2t.$$

Next,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right).$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]\right) = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}e^{-2s}\right] = u_2(t)\left(\frac{1}{4}(t-2) - \frac{1}{8}\sin 2(t-2)\right).$$

The solution is

$$y(t) = \cos 2t + \frac{1}{2}\sin 2t + u_2(t)\left(\frac{1}{4}(t-2) - \frac{1}{8}\sin 2(t-2)\right). \square$$

Example. Compute the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-3s-4}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-3s-4}\right] = \mathcal{L}^{-1}\left[\frac{1}{5}\left(\frac{1}{s-4} - \frac{1}{s+1}\right)\right] = \frac{1}{5}\left(e^{4t} - e^t\right). \square$$

Example. Compute the transform $\mathcal{L}\left[f(t)\right]$, where

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \\ t^2 + 2 & \text{if } t \ge 2 \end{cases}$$

$$f(t) = t \cdot (u_0 - u_1) + 1 \cdot (u_1 - u_2) + (t^2 + 2)u_2 = tu_0 + (1 - t)u_1 + (t^2 + 1)u_2.$$

Note that

$$(1 - t)u_1 = -(t - 1)u_1 \quad \text{and} \quad (t^2 + 1)u_2 = \left[(t - 2)^2 + 4(t - 2) + 5\right] u_2.$$

Therefore, using the step function shift,

$$F(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} + \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s}\right) e^{-2s}. \square$$