Math 228
3-19-1994

Review Problems for Test 2 (Class)

Example. Compute $\mathcal{L}\left((\sin t)u_\pi(t) + tu_4(t)\right)$.

Do the terms separately. For $(\sin t)u_\pi(t)$, I need to write $\sin t$ in terms of $t-\pi$.

$$\sin t = \sin\left[(t-\pi)+\pi\right] = \sin(t-\pi)\cos\pi + \cos(t-\pi)\sin\pi = -\sin(t-\pi).$$

Since $\mathcal{L}(-\sin t) = -\dfrac{1}{s^2+1}$,

$$\mathcal{L}\left((\sin t)u_\pi(t)\right) = -\frac{e^{-\pi s}}{s^2+1}.$$

Likewise, to do $tu_4(t)$ I need to write t in terms of $t-4$. $t = (t-4)+4$, and $\mathcal{L}(t+4) = \dfrac{1}{s^2} + \dfrac{4}{s}$. Therefore,

$$\mathcal{L}\left(tu_4(t)\right) = \left(\frac{1}{s^2} + \frac{4}{s}\right)e^{-4s}.$$

All together,

$$\mathcal{L}\left((\sin t)u_\pi(t) + tu_4(t)\right) = -\frac{e^{-\pi s}}{s^2+1} + \left(\frac{1}{s^2} + \frac{4}{s}\right)e^{-4s}.\ \square$$

Example. $y'' - 2y' + y = \dfrac{e^x}{x}$

The homogeneous solutions are $y_1 = e^x$ and $y_2 = xe^x$. The solution will have the form

$$y = u_1 e^x + u_2 x e^x.$$

The variation of parameters equation is

$$\begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{e^x}{x} \end{bmatrix}.$$

Invert the coefficient matrix on the left and put it on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{-2x} \begin{bmatrix} xe^x + e^x & -xe^x \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{e^x}{x} \end{bmatrix} = \begin{bmatrix} -1 \\ \dfrac{1}{x} \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -1 \quad , u_1 = -x + c_1,$$
$$u_2' = \frac{1}{x}, \quad u_2 = \ln|x| + c_2.$$

The general solution is

$$y = -xe^x + xe^x \ln|x| + c_1 e^x + c_2 x e^x.\ \square$$

Example. Compute $\mathcal{L}^{-1}\left(\dfrac{s-4}{s^2+s-2}e^{-3s}\right)$.

This is a step function shift with $c = 3$. I need to invert $\dfrac{s-4}{s^2+s-2}$. By partial fractions,

$$\frac{s-4}{s^2+s-2} = \frac{2}{s+2} - \frac{1}{s-1}.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{s-4}{s^2+s-2}\right) = 2e^{-2t} - e^t.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{s-4}{s^2+s-2}e^{-3s}\right) = u_3(t)\left[2e^{-2(t-3)} - e^{t-3}\right]. \square$$

Example. Find the general solution to $xy'' + 2y' + xy = 0$, given that $y_1 = \dfrac{\sin x}{x}$ is a solution.

Use reduction of order. The solution should have the form $y = v \cdot \dfrac{\sin x}{x}$, and I need to determine v.

Here's a trick that will simplify the algebra. I'll work with $y = vy_1$ (instead of the explicit expression for y_1). I'll plug in $y_1 = \dfrac{\sin x}{x}$ *after* the smoke has cleared.

Compute the derivatives:

$$y' = v'y_1 + vy_1', \quad y'' = v''y_1 + 2v'y_1' + vy_1''.$$

Plug them into the equation:

$$\begin{array}{rcccccc} xy'' + 2y' + xy & = & xv''y_1 & + & 2xv'y_1' & + & xvy_1'' \\ & & & + & 2v'y_1 & + & 2vy_1' \\ & & & & & + & xvy_1 \\ & = & xv''y_1 & + & (2xy_1' + 2y_1)v' & + & (xy_1'' + 2y_1' + xy_1)v \\ & = & xv''y_1 & + & (2xy_1' + 2y_1)v'. & & \end{array}$$

($xy_1'' + 2y_1' + xy_1 = 0$ because y_1 is a solution.)

Let $u = v'$, so $u' = v''$. Then

$$xu'y_1 + 2(xy_1' + y_1)u = 0.$$

Now I'll put in $y_1 = \dfrac{\sin x}{x}$. First,

$$y_1' = -\frac{\sin x}{x^2} + \frac{\cos x}{x}.$$

So the equation is

$$xu' \cdot \frac{\sin x}{x} + 2\left[x\left(-\frac{\sin x}{x^2} + \frac{\cos x}{x}\right) + \frac{\sin x}{x}\right]u = 0.$$

Simplify:

$$u' \sin x + 2u \cos x = 0.$$

Separate variables:

$$\int \frac{du}{u} = \int -\frac{2\cos x}{\sin x}\,dx, \quad \ln|u| = -2\ln|\sin x| + C, \quad u = \frac{C_0}{(\sin x)^2}.$$

Put v back:

$$v' = C_0(\csc x)^2, \quad v = -C_0 \cot x + C_1.$$

Finally,

$$y = v \cdot \frac{\sin x}{x} = C_2 \frac{\cos x}{x} + C_1 \frac{\sin x}{x}. \quad \square$$

Example. Find the general solution to

$$y'' + \frac{2}{x} y' + y = \frac{1}{x}.$$

The equation in the last problem was $xy'' + 2y' + xy = 0$. If I divide it by x, I get $y'' + \frac{2}{x} y' + y = 0$, which has the same left side as the equation I'm trying to solve.

Therefore, I know from the previous problem that $\frac{\cos x}{x}$ and $\frac{\sin x}{x}$ are independent homogeneous solutions for $y'' + \frac{2}{x} y' + y = \frac{1}{x}$. I'll solve the equation using variation of parameters.

The solution should have the form

$$y = u_1 \cdot \frac{\cos x}{x} + u_2 \cdot \frac{\sin x}{x}.$$

Set up the variation of parameters equation:

$$\begin{bmatrix} \frac{\cos x}{x} & \frac{\sin x}{x} \\ -\frac{\cos x}{x^2} - \frac{\sin x}{x} & -\frac{\sin x}{x^2} + \frac{\cos x}{x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{x} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = x^2 \cdot \begin{bmatrix} -\frac{\sin x}{x^2} + \frac{\cos x}{x} & -\frac{\sin x}{x} \\ \frac{\cos x}{x^2} + \frac{\sin x}{x} & \frac{\cos x}{x} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{x} \end{bmatrix} = \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix}.$$

$u_1' = -\sin x$ so $u_1 = \cos x + c_1$. $u_2' = \cos x$ so $u_2 = \sin x + c_2$. The general solution is

$$y = c_1 \frac{\cos x}{x} + c_2 \frac{\sin x}{x} + \frac{1}{x}. \quad \square$$

Example. Find the general solution to

$$y'' + 9y = -30 \sin 3x.$$

Write the equation as $(D^2 + 9)y = -30 \sin 3x$. The homogeneous solution is

$$y_h = c_1 \cos 3x + c_2 \sin 3x.$$

$-30 \sin 3x$ is killed by $D^2 + 9$. Hence, a particular solution should satisfy

$$(D^2 + 9)^2 y_p = (D^2 + 9)(-30 \sin 3x) = 0.$$

Therefore,

$$y_p = A \cos 3x + B \sin 3x + Cx \cos 3x + Dx \sin 3x.$$

$A \cos 3x$ and $B \sin 3x$ are homogeneous solutions — throw them out. I am left with

$$y_p = Cx \cos 3x + Dx \sin 3x.$$

Differentiate:

$$y_p' = C\cos 3x - 3Cx\sin 3x + D\sin 3x + 3Dx\cos 3x,$$

$$y_p'' = -6C\sin 3x - 9Cx\cos 3x + 6D\cos 3x - 9Dx\sin 3x.$$

Plug these into the original equation:

$$y_p'' + 9y_p = -6C\sin 3x + 6D\cos 3x = -30\sin 3x.$$

Equate corresponding coefficients. I get $C = 5$, $D = 0$. Therefore, $y_p = 5x\cos 3x$. The general solution is

$$y = c_1\cos 3x + c_2\sin 3x + 5x\cos 3x.$$

It's also possible to do this problem using complex exponentials; you might want to redo the problem that way and see if you think it's easier. □