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4-25-1994

Review Session for Test 3.

1. Solve the initial value problem

$$\vec{x}\,' = \begin{bmatrix} 3 & -4 \\ 0 & 3 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}:$$

(a) Using generalized eigenvectors.

Since the matrix is triangular, the eigenvalues are the diagonal entries. The only eigenvalue is $\lambda = 3$, a double root.

$$A - 3I = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}$$

$\langle a, b\rangle = \langle 1, 0\rangle$ is an eigenvector.
Next, I want to find a generalized eigenvector $\langle c, d\rangle$. It must satisfy

$$\begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

I get $-4d = 1$, so $d = -\dfrac{1}{4}$. Hence,

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c \\ -\frac{1}{4} \end{bmatrix}.$$

Take $c = 0$; I get $\left\langle 0, -\dfrac{1}{4}\right\rangle$.
Therefore, the general solution is

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(te^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} \right).$$

Plug in the initial condition:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}.$$

This gives $c_1 = 1$, $c_2 = 4$. The solution is

$$\vec{x} = e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \left(te^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} \right) = \begin{bmatrix} e^{3t} + 4te^{3t} \\ -e^{3t} \end{bmatrix}. \square$$

(b) Using the matrix exponential.

First, I'll compute the a_k's:

$$a_1 = e^{3t}, \quad a_2 = e^{3t} \star e^{3t} = \int_0^t e^{3(t-u)} e^{3u}\, du = e^{3t} \int_0^t du = te^{3t}.$$

Next, I'll compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A - 3I = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$e^{At} = e^{3t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{3t}\begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{3t} & -4te^{3t} \\ 0 & e^{3t} \end{bmatrix}.$$

The solution is

$$\vec{x} = \begin{bmatrix} e^{3t} & -4te^{3t} \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad \square$$

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2. Find the series solution about $x_0 = 0$ for

$$(x^2 + 1)y'' + 2xy' + 2y = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

In the first, third, and fourth sums, let $n = k - 2$. When $n = 0$, $k = 2$, so the sums become

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{k=2}^{\infty}(k-2)(k-3)a_{k-2}x^{k-2} = \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2},$$

$$2\sum_{n=0}^{\infty} n a_n x^n = 2\sum_{k=2}^{\infty}(k-2)a_{k-2}x^{k-2} = 2\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2},$$

$$2\sum_{n=0}^{\infty} a_n x^n = 2\sum_{k=2}^{\infty} a_{k-2}x^{k-2} = 2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

In each case, I obtained the second equality by replacing the dummy variable k with n.
The differential equation is now

$$\sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2} + 2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

The sum of the coefficients for each power of x on the left must be 0. For $n = 0$ and $n = 1$, I get $0 = 0$.
For $n \geq 2$, I get

$$(n-2)(n-3)a_{n-2}a_{n-2} + n(n-1)a_n + 2(n-2)a_{n-2} + 2a_{n-2} = 0, \quad a_n = -\frac{n^2 - 3n + 4}{n(n-1)}a_{n-2}.$$

Plug in even numbers for n:

$$\begin{aligned} a_2 &= -\frac{2}{2 \cdot 1}a_0, \\ a_4 &= -\frac{8}{4 \cdot 3}a_2, \\ &\vdots \\ a_{2n} &= -\frac{4n^2 - 6n + 4}{2n(2n-1)}a_{2n-2}. \end{aligned}$$

Multiply the equations together — that is, multiply the left sides, multiply the right sides, then set the two products equal:

$$a_2 a_4 \cdots a_{2n} = \left(-\frac{2}{2\cdot 1}\right)\left(-\frac{8}{4\cdot 3}\right)\cdots\left(-\frac{4n^2-6n+4}{2n(2n-1)}\right) a_0 a_2 \cdots a_{2n-2}.$$

Cancel the common terms a_2, a_4, ..., a_{2n-2} from both sides:

$$a_{2n} = \left(-\frac{2}{2\cdot 1}\right)\left(-\frac{8}{4\cdot 3}\right)\cdots\left(-\frac{4n^2-6n+4}{2n(2n-1)}\right) a_0.$$

There are n negative signs, giving a factor of $(-1)^n$. The denominators of the fractions multiply to $(2n)!$. So

$$a_{2n} = (-1)^n \frac{2\cdot 8\cdots(4n^2-6n+4)}{(2n)!} a_0.$$

Likewise, plug in odd numbers for n:

$$\begin{aligned} a_3 &= -\frac{4}{3\cdot 2} a_1, \\ a_5 &= -\frac{14}{5\cdot 4} a_3, \\ &\vdots \\ a_{2n+1} &= -\frac{4n^2-2n+2}{(2n+1)(2n)} a_{2n-1}. \end{aligned}$$

Multiply the equations together — that is, multiply the left sides, multiply the right sides, then set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \left(-\frac{4}{3\cdot 2}\right)\left(-\frac{14}{5\cdot 4}\right)\cdots\left(-\frac{4n^2-2n+2}{(2n+1)(2n)}\right) a_1 a_3 \cdots a_{2n-1}.$$

Cancel the common factors a_3, a_5, ..., a_{2n-1} from both sides:

$$a_{2n+1} = \left(-\frac{4}{3\cdot 2}\right)\left(-\frac{14}{5\cdot 4}\right)\cdots\left(-\frac{4n^2-2n+2}{(2n+1)(2n)}\right) a_1.$$

There are n negative signs, giving a factor of $(-1)^n$. The denominators of the fractions multiply to $(2n)!$. So

$$a_{2n+1} = (-1)^n \frac{4\cdot 14\cdots(4n^2-2n+2)}{(2n+1)!} a_1.$$

The solution is

$$y = a_0\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{2\cdot 8\cdots(4n^2-6n+4)}{(2n)!} x^{2n}\right) + a_1\left(x + \sum_{n=1}^{\infty}(-1)^n \frac{4\cdot 14\cdots(4n^2-2n+2)}{(2n+1)!} x^{2n+1}\right). \square$$

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3. Locate the critical points of the almost linear system

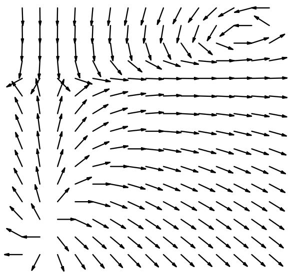
$$\begin{aligned} \frac{dx}{dt} &= x(2-y), \\ \frac{dy}{dt} &= y(x-y-6). \end{aligned}$$

For each critical point, find the corresponding linear system, and find the eigenvalues and eigenvectors of the corresponding linear system. Classify each critical point, and determine whether the point is stable, asymptotically stable, or unstable.

First, I'll find the critical points:

$$\begin{array}{ccccc}
 & & x(2-y)=0 & & \\
 & \swarrow & & \searrow & \\
x=0 & & & & y=2 \\
y(x-y-6)=0 & & & & y(x-y-6)=0 \\
y(-y-6)=0 & & & & 2(x-8)=0 \\
\downarrow & \searrow & & & x=8 \\
y=0 & & y=-6 & & (8,2) \\
(0,0) & & (0,-6) & &
\end{array}$$

The critical points are $(0,0)$, $(0,-6)$, and $(8,2)$.



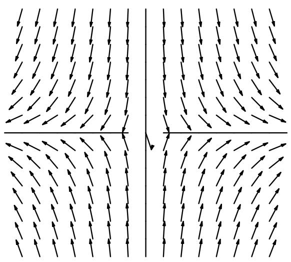
The matrix of partial derivatives is

$$A = \begin{bmatrix} 2-y & -x \\ y & x-2y-6 \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix}.$$

$\det = -12$, so the point is a saddle, and it's unstable.

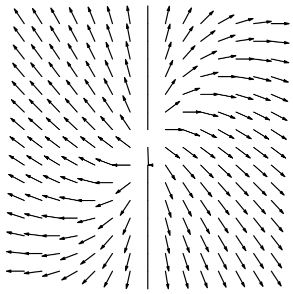


The eigenvalues are 2 and -6; the corresponding eigenvectors are $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$.

At $(0, -6)$,

$$A = \begin{bmatrix} 8 & 0 \\ -6 & 6 \end{bmatrix}.$$

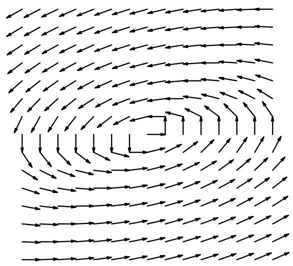
$\det = 48$ and $\operatorname{tr} = 14$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an unstable improper node.



The eigenvalues are 8 and 6; the corresponding eigenvectors are $\langle 1, -3\rangle$ and $\langle 0, 1\rangle$.
At $(8, 2)$,

$$A = \begin{bmatrix} 0 & -8 \\ 2 & -2 \end{bmatrix}.$$

$\det = 16$, $\operatorname{tr} = -2$, so $\det > \dfrac{1}{4}\operatorname{tr}^2$. The point is an asymptotically stable spiral.



The eigenvalues are $-1 \pm i\sqrt{15}$; the corresponding eigenvectors are $\langle 8, 1 - i\sqrt{15}\rangle$ and $\langle 81 + i\sqrt{15}\rangle$. $\square$