Series Solutions - Ordinary Points.

Consider a second-order differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

For simplicity, p, q, and r will be polynomials in most of the examples.

I'll construct a Taylor expansion for the solution:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

I'll assume that the expansion point x_0 is an **ordinary point** — that is, a point which is *not* a root of $p(x)$. (If $p(x_0) = 0$, x_0 is a **singular point**, and obtaining a series solution is more complicated.)

The problem is to determine the coefficients a_n. The idea is simple: Plug the series into the differential equation and look at the coefficients of each power x^n. In most cases, you can obtain a **recursion formula** for the a_n, which allows you to generate as many terms of the series as you need. In an application, you take as many terms of the series as you need to approximate the solution to the desired degree of accuracy.

Of course, with any infinite series there is the issue of convergence. I'll consider that problem later on.

Example. Find a series solution for $y'' + y = 0$ about the point $x_0 = 0$.

This is a constant coefficient equation, and the solution should be $y = c_1 \cos x + c_2 \sin x$.

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(The powers are x^n because $x_0 = 0$.)

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Plug the sums into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

In order to compare coefficients, I'd like all of the sums to have the "same" power of x. A rule of thumb is to make all of the powers of x the same as the *smallest* power that occurs — in this case, x^{n-2}. Therefore, I need to make a substitution in the second sum.

In the second sum, let $n = k - 2$. When $n = 0$, $k = 2$, so

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=2}^{\infty} a_{k-2} x^{k-2} = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

The second equality follows by replacing the dummy variable k with n.

(It's possible to do this substitution in one step, without introducing k — in fact, there are books in which the author says: "Let $n = n - 2$". In that case, you have to keep straight which is the "old" n and which is the "new" n, for instance when you let $n = 0$. For the sake of clarity, I'll write out the substitution the long way, as I did above.)

Now the differential equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

Now I'll look at the coefficients of x^0, x^1, x^2, and so on. Since I'm just looking at the coefficients, and since the powers of x are the same in the two sums, I just ignore the x's.

For $n = 0$ or $n = 1$, the equation says $0 = 0$. Try $n = 2$:

$$2a_2 + a_0 = 0, \qquad \text{so} \quad a_2 = -\frac{1}{2}a_0.$$

In fact, for $n \geq 2$,

$$n(n-1)a_n + a_{n-2} = 0, \quad \text{so} \quad a_n = -\frac{1}{n(n-1)}a_{n-2}.$$

This recursion formula gives a_n in terms of the preceding a's (in fact, in terms of a_{n-2}. a_3 comes out in terms of a_1, a_4 in terms of a_2, a_{117} in terms of a_{115}, and so on.

In cases where a_n depends on *only one of* a_{n-2} or a_{n-1}, you can use the following trick to find a more explicit formula for a_n. Since $a_n = -\dfrac{1}{n(n-1)}a_{n-2}$ involves a_{n-2}, I'll consider *two* cases: n is even or n is odd.

Substitute even numbers for n:

$$\begin{aligned} a_2 &= -\frac{1}{2}a_0, \\ a_4 &= -\frac{1}{4\cdot 3}a_2, \\ a_6 &= -\frac{1}{6\cdot 5}a_4, \\ &\vdots \\ a_{2n} &= -\frac{1}{2n(2n-1)}a_{2n-2}. \end{aligned}$$

Multiply these equations together — that is, multiply the left sides together, multiply the right sides together, then set the two products equal:

$$a_2a_4a_6\cdots a_{2n} = \left(-\frac{1}{2}\right)\left(-\frac{1}{4\cdot 3}\right)\left(-\frac{1}{6\cdot 5}\right)\cdots\left(-\frac{1}{2n(2n-1)}\right)a_0a_2a_4\cdots a_{2n-2}.$$

Cancel the common factors a_2, a_4, $\ldots$, a_{2n-2} from both sides:

$$a_{2n} = \left(-\frac{1}{2}\right)\left(-\frac{1}{4\cdot 3}\right)\left(-\frac{1}{6\cdot 5}\right)\cdots\left(-\frac{1}{2n(2n-1)}\right)a_0.$$

This gives *any even* a in terms of a_0. But I can simplify this formula further. For one thing, these are n negative signs there, which gives $(-1)^n$ all together. Next, notice that the numbers on the bottoms of the fractions account for all the numbers from 1 to $2n$. Hence, their product is $(2n)!$. So the formula is

$$a_{2n} = (-1)^n\frac{1}{(2n)!}a_0.$$

The same trick works with the odd a's. Substitute odd numbers for n:

$$\begin{aligned} a_3 &= -\frac{1}{3\cdot 2}a_1, \\ a_5 &= -\frac{1}{5\cdot 4}a_3, \\ &\vdots \\ a_{2n+1} &= -\frac{1}{(2n+1)(2n)}a_{2n-1}. \end{aligned}$$

Multiply these equations together — that is, multiply the left sides together, multiply the right sides together, then set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \left(-\frac{1}{3\cdot 2}\right)\left(-\frac{1}{5\cdot 4}\right)\cdots\left(-\frac{1}{(2n+1)(2n)}\right) a_1 a_3 \cdots a_{2n-1}.$$

Cancel the common factors a_3, a_5, ..., a_{2n_1} from both sides:

$$a_{2n+1} = \left(-\frac{1}{3\cdot 2}\right)\left(-\frac{1}{5\cdot 4}\right)\cdots\left(-\frac{1}{(2n+1)(2n)}\right) a_1.$$

As before, there are n negative signs, which gives $(-1)^n$. The numbers on the bottoms of the fractions multiply to $(2n+1)!$. Thus,

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)!} a_1.$$

Now I put the coefficients back into the series. Since I have separate formulas for the odd terms and the even terms, I'll write them separately. The solution is

$$y = a_0 + \sum_{n=1}^{\infty} a_{2n}x^{2n} + a_1 x + \sum_{n=1}^{\infty} a_{2n+1}x^{2n+1} = a_0 + \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n)!}a_0 x^{2n} + a_1 x + \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)!}a_1 x^{2n+1} =$$

$$a_0\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}\right) + a_1\left(x + \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n+1}\right).$$

The expression in the first set of parentheses is the series for $\cos x$. The expression in the second set of parentheses is the series for $\sin x$. So the solution comes out to

$$y = a_0 \cos x + a_1 \sin x.$$

Notice that the first two coefficients a_0 and a_1 play the role of the (expected) two arbitrary constants. □

Example. Find a series solution for $y'' + 4xy' + 5y = 0$ about the point $x_0 = 0$.

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(The powers are x^n because $x_0 = 0$.)
The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Plug the sums into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} n a_n x^n + 5\sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second and third series, substitute $n = k - 2$. When $n = 0$, $k = 2$. The two series become

$$4\sum_{n=0}^{\infty} n a_n x^n = 4\sum_{k=2}^{\infty}(k-2)a_{k-2}x^{k-2} = 4\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2},$$

$$5\sum_{n=0}^{\infty} a_n x^n = 5\sum_{k=2}^{\infty} a_{k-2}x^{k-2} = 5\sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

In each case, the second equalities follow by replacing the dummy variable k with n.
The equation is now

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2} + 5\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

Look at the coefficients of each power of x on the left. For $n = 0$ or $n = 1$, I get $0 = 0$. For $n \geq 2$,

$$n(n-1)a_n + 4(n-2)a_{n-2} + 5a_{n-2} = 0,$$

$$a_n = -\frac{4n-3}{n(n-1)}a_{n-2}.$$

Substitute even numbers for n:

$$\begin{aligned} a_2 &= -\frac{5}{2}a_0, \\ a_4 &= -\frac{13}{4\cdot 3}a_2, \\ &\vdots \\ a_{2n} &= -\frac{8n-3}{2n(2n-1)}a_{2n-2}. \end{aligned}$$

Multiply these equations together — that is, multiply the left sides together, multiply the right sides together, then set the two products equal:

$$a_2 a_4 \cdots a_{2n} = \left(-\frac{5}{2}\right)\left(-\frac{13}{4\cdot 3}\right)\left(-\frac{8n-3}{2n(2n-1)}\right) a_0 a_2 a_4 \cdots a_{2n-2}.$$

Cancel the common factors a_2, a_4, $\ldots$, a_{2n-2} from both sides:

$$a_{2n} = \left(-\frac{5}{2}\right)\left(-\frac{13}{4\cdot 3}\right)\left(-\frac{8n-3}{2n(2n-1)}\right) a_0.$$

There are n negative signs, so they multiply to $(-1)^n$. Notice that the numbers on the bottoms of the fractions account for all the numbers from 1 to $2n$. Hence, their product is $(2n)!$. So the formula is

$$a_{2n} = (-1)^n \frac{5\cdot 13\cdots(8n-3)}{(2n)!}a_0.$$

The same trick works with the odd a's. Substitute odd numbers for n:

$$\begin{aligned} a_3 &= -\frac{9}{3\cdot 2}a_1, \\ a_5 &= -\frac{17}{5\cdot 4}a_3, \\ &\vdots \\ a_{2n+1} &= -\frac{8n+1}{(2n+1)(2n)}a_{2n-1}. \end{aligned}$$

Multiply these equations together — that is, multiply the left sides together, multiply the right sides together, then set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \left(-\frac{9}{3\cdot 2}\right)\left(-\frac{17}{5\cdot 4}\right)\cdots\left(\frac{8n+1}{(2n+1)(2n)}\right) a_1 a_3 \cdots a_{2n-1}.$$

Cancel the common factors a_3, a_5, ..., a_{2n_1} from both sides:

$$a_{2n+1} = \left(-\frac{9}{3\cdot 2}\right)\left(-\frac{17}{5\cdot 4}\right)\cdots\left(\frac{8n+1}{(2n+1)(2n)}\right) a_1.$$

As before, there are n negative signs, which gives $(-1)^n$. The numbers on the bottoms of the fractions multiply to $(2n+1)!$. Thus,

$$a_{2n+1} = (-1)^n \frac{9\cdot 17\cdots(8n+1)}{(2n+1)!} a_1.$$

The series solution is

$$y = a_0\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{5\cdot 13\cdots(8n-3)}{(2n)!} x^{2n}\right) + a_1\left(x + \sum_{n=1}^{\infty}(-1)^n \frac{9\cdot 17\cdots(8n+1)}{(2n+1)!} x^{2n+1}\right).$$

In many cases, it isn't possible to get an explicit expression for a_n in terms of a_0 and a_1. This is the case in the next example. $\square$

Example. Find a series solution for $y'' + x^2 y' - 2y = 0$ about $x_0 = 0$.

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(The powers are x^n because $x_0 = 0$.)
The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Plug the sums into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second sum, let $n+1 = k-2$, so $n = k-3$. When $n = 0$, $k = 3$. The sum becomes

$$\sum_{n=0}^{\infty} n a_n x^{n+1} = \sum_{k=3}^{\infty}(k-3) a_{k-3} x^{k-2} = \sum_{n=3}^{\infty}(n-3) a_{n-3} x^{n-2}.$$

The second equality follows by replacing the dummy variable k with n.
In the third sum, substitute $n = k-2$. When $n = 0$, $k = 2$. The series becomes

$$-2\sum_{n=0}^{\infty} a_n x^n = -2\sum_{k=2}^{\infty} a_{k-2} x^{k-2} = -2\sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

The second equality follows by replacing the dummy variable k with n.
The differential equation now reads

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=3}^{\infty}(n-3) a_{n-3} x^{n-2} - 2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0.$$

Look at the coefficients of each power of x on the left. For $n = 0$ or $n = 1$, I get $0 = 0$. For $n = 2$, I get

$$2a_2 - 2a_0 = 0, \quad \text{so} \quad a_2 = a_0.$$

For $n \ge 3$,

$$n(n-1)a_n + (n-3)a_{n-3} - 2a_{n-2} = 0,$$

$$a_n = \frac{-(n-3)a_{n-3} + 2a_{n-2}}{n(n-1)}.$$

Since a_n depends on *both* a_{n-2} and a_{n-3}, I can't use the trick I used in the previous examples. This is really as far as I can go.

However, I *can* use *Mathematica* to express the higher order coefficients in terms of a_0 and a_1. Here's how. First, I need to define a_0, a_1, and a_2:

```
a[0] = a0; a[1] = a1; a[2] = a0;
```

(I need a_2 defined because the recursion formula only works for $n \ge 3$.) Next, enter the recursion formula:

```
a[n_] := a[n] = Expand[(-(n - 3)a[n-3] + 2a[n-2])/(n(n - 1))]
```

(The "`a[n_] := a[n] =`" defines a **memo function**; it's useful when you're defining a function recursively. At each stage, *Mathematica* computes the current a_n based on the previous a's. The definition above forces *Mathematica* to "remember" the previous a's as it computes them, so it doesn't recompute a value each time it needs it.)

Now (for example) `a[10]` produces a_{10} in terms of a_0 and a_1. Here's the solution:

$$y = a_0\left(1 + x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 + \cdots\right) + a_1\left(x + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots\right).$$

Here's an easy way to produce the a_0 and a_1 parts using *Mathematica.* First, I construct the series through (say) x^{10}:

```
sum10 = Sum[a[n] x^n, {n, 0, 10}]
```

Get the a_0 part by setting $a_1 = 0$, and the a_1 part by setting $a_0 = 0$:

```
sum10 /.  (a1->0)
```

```
sum10 /.  (a0->0)
```

Try it yourself! □

Example. Find a series solution for $y'' + (x+1)y' - 2y = 0$ about $x_0 = 1$.

This means that I want the solution to look like

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n.$$

Let $u = x - 1$. Then $x + 1 = u + 2$. Moreover, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2y}{du^2}.$$

So the equation becomes

$$y'' + (u+2)y' - 2y = 0.$$

(I'm abusing notation — this y'' is really $\dfrac{d^2y}{du^2}$ and y' is really $\dfrac{dy}{du}$. But the previous remark about the derivatives shows that there's no harm.)

I'll find a series solution to this equation about $u_0 = 0$:

$$y = \sum_{n=0}^{\infty} a_n u^n.$$

Then I'll put $u = x - 1$ to solve the original problem.
The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n u^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n u^{n-2}.$$

Plug the sums into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1) a_n u^{n-2} + \sum_{n=0}^{\infty} n a_n u^n + 2 \sum_{n=0}^{\infty} n a_n u^{n-1} - 2 \sum_{n=0}^{\infty} a_n u^n = 0.$$

In the second and fourth sums, set $n = k - 2$. When $n = 0$, $k = 2$. The sums become

$$\sum_{n=0}^{\infty} n a_n u^n = \sum_{k=2}^{\infty} (k-2) a_{k-2} u^{k-2} = \sum_{n=2}^{\infty} (n-2) a_{n-2} u^{n-2},$$

$$-2 \sum_{n=0}^{\infty} a_n u^n = -2 \sum_{k=2}^{\infty} a_{k-2} u^{k-2} = -2 \sum_{n=2}^{\infty} a_{n-2} u^{n-2}.$$

In each case, the second equality follows by replacing the dummy variable k with n.
In the third sum, set $n - 1 = k - 2$. Then $n = k - 1$, and when $n = 0$, $k = 1$. The sum becomes

$$2 \sum_{n=0}^{\infty} n a_n u^{n-1} = 2 \sum_{k=1}^{\infty} (k-1) a_{k-1} u^{k-2} = 2 \sum_{n=1}^{\infty} (n-1) a_{n-1} u^{n-2}.$$

The differential equation is now

$$\sum_{n=0}^{\infty} n(n-1) a_n u^{n-2} + \sum_{n=2}^{\infty} (n-2) a_{n-2} u^{n-2} + 2 \sum_{n=1}^{\infty} (n-1) a_{n-1} u^{n-2} - 2 \sum_{n=2}^{\infty} a_{n-2} u^{n-2} = 0.$$

Look at the coefficients of each power of x on the left. For $n = 0$ or $n = 1$, I get $0 = 0$. For $n \geq 2$,

$$n(n-1) a_n + (n-2) a_{n-2} + 2(n-1) a_{n-1} - 2 a_{n-2} = 0,$$

$$a_n = -\frac{(n-4) a_{n-2} + 2(n-1) a_{n-1}}{n(n-1)}.$$

I can use *Mathematica* to obtain as many coefficients as I want. The solution is

$$y = a_0 \left(1 + (x-1)^2 - \frac{2}{3}(x-1)^3 + \frac{1}{3}(x-1)^4 - \cdots\right) +$$

$$a_1 \left((x-1) - (x-1)^2 + \frac{5}{6}(x-1)^3 - \frac{5}{12}(x-1)^4 + \cdots\right). \quad \square$$

Example. Find the first three nonzero terms of the series solution about $x_0 = 0$ for the initial value problem

$$y'' + (\cos 2x) y' + (x^2 + 1) y = 0, \qquad y(0) = 1, y'(0) = 0.$$

With initial values given, I can find the coefficients of the Taylor expansion by repeated ifferentiation. Plug $x = 0$ into the equation:

$$y''(0) + (\cos 0)y'(0) + (1)y(0) = 0, \quad y''(0) = -1.$$

Differentiate the original equation:

$$y''' + (\cos 2x)y'' - (2\sin 2x)y' + 2xy + (x^2 + 1)y' = 0.$$

Plug in $x = 0$:

$$y'''(0) + (\cos 0)y''(0) - (2\sin 0)y'(0) + (0)y(0) + (1)y'(0) = 0, \quad y'''(0) = 1.$$

Notice that at each step the next derivative comes out in terms of the previous derivatives, *which I've already computed.* I can continue this procedure indefinitely.

The first few terms of the Taylor expansion are

$$y = 1 + 0 \cdot x + \frac{(-1)}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

Notice that in this problem I'm computing the *derivatives.* In the other examples, I was computing the a_n's, which were the whole coefficients, *including the factorials.* So in this problem, I had to remember to divide each term by $n!$ as I wrote down the series. $\square$

Example. Find the series solution through the 4-th degree term for the differential equation

$$y'' + \frac{1}{1 - x^3}y' + y = 0, \qquad x_0 = 0.$$

Here's a standard Taylor series:

$$\frac{1}{1 - u} = 1 + u + u^2 + u^3 + \cdots.$$

By setting $u = x^3$, I get

$$\frac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + \cdots.$$

I'll stop after x^6, since that will be enough for the desired accuracy. The equation is now

$$y'' + (1 + x^3 + x^6)y' + y = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

(The powers are x^n because $x_0 = 0$.)
The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Plug the sums into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n+2} + \sum_{n=0}^{\infty} n a_n x^{n+5} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second sum, set $n - 1 = k - 2$, so $n = k - 1$. When $n = 0$, $k = 1$. The sum becomes

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{k=1} (k-1) a_{k-1} x^{k-2} = \sum_{n=1} (n-1) a_{n-1} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
In the third sum, set $n + 2 = k - 2$, so $n = k - 4$. When $n = 0$, $k = 4$. The sum becomes

$$\sum_{n=0}^{\infty} n a_n x^{n+2} = \sum_{k=4} (k-4) a_{k-4} x^{k-2} = \sum_{n=4} (n-4) a_{n-4} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
In the fourth sum, set $n + 5 = k - 2$, so $n = k - 7$. When $n = 0$, $k = 7$. The sum becomes

$$\sum_{n=0}^{\infty} n a_n x^{n+5} = \sum_{k=7} (k-7) a_{k-7} x^{k-2} = \sum_{n=7} (n-7) a_{n-7} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
In the fifth sum, set $n = k - 2$. When $n = 0$, $k = 2$. The sum becomes

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=2} a_{k-2} x^{k-2} = \sum_{n=2} a_{n-2} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
The differential equation is now

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1} (n-1) a_{n-1} x^{n-2} + \sum_{n=4} (n-4) a_{n-4} x^{n-2} + \sum_{n=7} (n-7) a_{n-7} x^{n-2} + \sum_{n=2} a_{n-2} x^{n-2} = 0.$$

Look at the coefficients of each power of x on the left. For $n = 0$ or $n = 1$, I get $0 = 0$. For $n = 2$, I get

$$2a_2 + a_1 + a_0 = 0, \quad \text{so} \quad a_2 = -\frac{1}{2} a_0 - \frac{1}{2} a_1.$$

For $n = 3$,

$$6a_3 + 2a_2 + a_1 = 0, \quad \text{so} \quad a_3 = \frac{1}{6} a_0.$$

(I used $a_2 = -\frac{1}{2} a_0 - \frac{1}{2} a_1$ from the previous computation.)
For $n = 4$,

$$12a_4 + 3a_3 + a_2 = 0, \quad \text{so} \quad a_4 = \frac{1}{24} a_1.$$

I could continue in this way, but at some point I'd need to go back and include the x^9 that I omitted from the series expansion for $\dfrac{1}{1 - x^3}$. The series — so far — is

$$y = a_0 + a_1 x + \left(-\frac{1}{2} a_0 - \frac{1}{2} a_1\right) x^2 + \frac{1}{6} a_0 x^3 + \frac{1}{24} a_1 x^4 + \cdots . \ \square$$

Finally, I want to say a brief word about convergence. Consider the equation

$$p(x) y'' + q(x) y' + r(x) y = 0.$$

Suppose p, q, and r are polynomials. A series solution

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

will converge in a symmetric interval of radius r about x_0. r is the radius of a disk in the complex plane $\mathbb{C}$ which does not contain any root of $p(x)$.

Example. A series solution for $y'' + 4xy' + 5y = 0$ about any point x_0 will converge for all x. The coefficient of y'' is 1, which has no roots. □

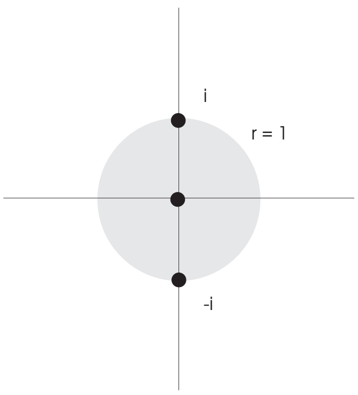
Example. Consider the equation $(x-1)(x-2)y'' + xy' + 2y = 0$. A series solution about $x_0 = 0$ will converge in the interval $-1 < x < 1$ ($r = 1$), because the nearest root of $(x-1)(x-2)$ is at $x = 1$.

A series solution about $x_0 = \dfrac{3}{2}$ will converge in the interval $1 < x < 2$ ($r = \dfrac{1}{2}$), because $x_0 = \dfrac{3}{2}$ is midway between the roots $x = 1$ and $x = 2$.

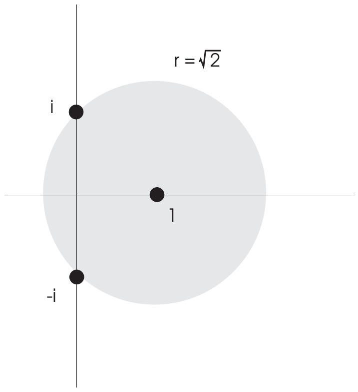
A series solution about $x_0 = 5$ will converge in the interval $2 < x < 8$ ($r = 3$), because the nearest root of $(x-1)(x-2)$ is at $x = 2$. □

Example. Consider the equation $(x^2+1)y'' + xy' + 2y = 0$. Notice that the roots of $x^2 + 1$ are at $\pm i$ in the complex plane.

A series solution about $x_0 = 0$ will converge in the interval $-1 < x < 1$ ($r = 1$), because the roots of $x^2 + 1$ are at $\pm i$, and they are each 1 unit away from $x_0 = 0$. Pictorially, if I draw circles centered at $x_0 = 0$, I will overrun the points $\pm i$ when the circles' radii pass $r = 1$.



A series solution about $x_0 = 1$ will converge in the interval $1 - \sqrt{2} < x < 1 + \sqrt{2}$ ($r = \sqrt{2}$). This time, if I draw circles centered at $x_0 = 1$, I will overrun the points $\pm i$ when the circles' radii pass $r = \sqrt{2}$.



$\square$