3-30-1994

Linear systems - complex roots.

If $\vec{v}$ is a vector, $\vec{v}^*$ will denote the complex conjugate of $\vec{v}$. For example,

$$\langle 1+2i, 3\rangle^* = \langle 1-2i, 3\rangle.$$

Complex eigenvalues of a real matrix occur in conjugate pairs $a \pm bi$. Suppose $\vec{v}$ is an eigenvector for $a+bi$. Then

$$(A - (a+bi)I)\,\vec{v} = 0.$$

Take the conjugate of both sides:

$$(A - (a-bi)I)\,\vec{v}^* = 0.$$

This shows that $\vec{v}^*$ is an eigenvector for $a - bi$.

$$\boxed{\text{If } \vec{v} \text{ is an eigenvector for } a+bi, \text{ then } \vec{v}^* \text{ is an eigenvector for } a-bi.}$$

Example.

$$A = \begin{bmatrix} 5 & -2 \\ 5 & -1 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} 5-\lambda & -2 \\ 5 & -1-\lambda \end{vmatrix} = (\lambda-5)(\lambda+1)+10 = \lambda^2 - 4\lambda + 5.$$

The eigenvalues are $\lambda = 2 \pm i$.
Consider $\lambda = 2+i$:

$$A - (2+i)I = \begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -\frac{1}{5}(3+i) \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \frac{1}{5}(3+i)b = 0$, or $a = \frac{1}{5}(3+i)b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(3+i)b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{5}(3+i) \\ 1 \end{bmatrix}.$$

Take $b = 5$. The eigenvector is $\langle 3+i, 5\rangle$.
By the result above, I don't need to do the computation for the eigenvalue $2-i$:

$$\langle 3+i, 5\rangle^* = \langle 3-i, 5\rangle$$

is an eigenvector for $2-i$. $\square$

Suppose that the linear system

$$\vec{x}\,' = Ax$$

has conjugate complex eigenvalues λ, λ^* with eigenvectors $\vec{v}$, $\vec{v}^{\,*}$, respectively. This yields solutions

$$e^{\lambda t}\vec{v}, \qquad e^{\lambda^* t}\vec{v}^{\,*}.$$

If $a + bi$ is a complex number,

$$\operatorname{re}(a+bi) = a = \frac{1}{2}\left((a+bi)+(a-bi)\right) = \frac{1}{2}\left((a+bi)+(a+bi)^*\right),$$

$$\operatorname{im}(a+bi) = b = \frac{1}{2}\left((a+bi)-(a-bi)\right) = \frac{1}{2}\left((a+bi)-(a+bi)^*\right).$$

I'll apply this to $e^{\lambda t}\vec{v}$, using the fact that

$$\left(e^{\lambda t}\vec{v}\right)^* = e^{\lambda^* t}\vec{v}^{\,*}.$$

Then

$$\operatorname{re}\left(e^{\lambda t}\vec{v}\right) = \frac{1}{2}\left(e^{\lambda t}\vec{v} + e^{\lambda^* t}\vec{v}^{\,*}\right),$$

$$\operatorname{im}\left(e^{\lambda t}\vec{v}\right) = \frac{1}{2}\left(e^{\lambda t}\vec{v} - e^{\lambda^* t}\vec{v}^{\,*}\right).$$

The point is that since the terms on the right are independent solutions, so are the terms on the left. The terms on the left, however, are *real* solutions. Here is what this means.

> If a linear system has a pair of complex conjugate eigenvalues, find the eigenvector solution for *one* of them (the "$e^{\lambda t}\vec{v}$" above). Then take the real and imaginary parts to obtain two independent real solutions.

Example. Solve the system

$$x_1' = x_1 - x_2,$$

$$x_2' = x_1 + x_2.$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues are $\lambda = 1 \pm i$.
Consider $\lambda = 1 + i$:

$$A - (1+i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - bi = 0$, so $a = bi$. The eigenvectors are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} bi \\ b \end{bmatrix} = b\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Take $b = 1$. This yields the eigenvector $\langle i, 1\rangle$.

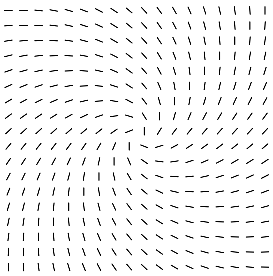
Write down the complex solution

$$e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^t e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^t(\cos t + i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} = e^t\begin{bmatrix} -\sin t & + & i\cos t \\ \cos t & + & i\sin t \end{bmatrix}.$$

Take the real and imaginary parts:

$$\operatorname{re} e^t\begin{bmatrix} -\sin t & + & i\cos t \\ \cos t & + & i\sin t \end{bmatrix} = e^t\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix},$$

$$\operatorname{im} e^t\begin{bmatrix} -\sin t & + & i\cos t \\ \cos t & + & i\sin t \end{bmatrix} = e^t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$



The general solution is

$$\vec{x} = c_1 e^t\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \square$$