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Undetermined coefficients.

A **nonhomogeneous linear equation with constant coefficients** is an equation of the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x).$$

The a_i's are constants, and as usual $y^{(n)}$ means the n-th derivative of y with respect to x.

You know that the corresponding homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

may be solved by constructing the **characteristic equation**

$$a_n m^n + \cdots + a_1 m + a_0 = 0$$

and finding the roots.

The method of **undetermined coefficients** constructs the general solution to the nonhomogeneous equation *if $f(x)$ itself is a solution to a linear homogeneous constant coefficient equation.*

This means that $f(x)$ can contain terms of the form

$$(b_m x^m + \cdots + b_1 x + b_0)e^{ax} \left\{ \begin{matrix} \cos bx \\ \sin bx \end{matrix} \right\}.$$

Here's the idea. Write the equation in operator form:

$$(a_n D^n + \cdots + a_1 D + a_0)y = f(x).$$

First, I'll find the general solution y_h to the homogeneous equation. (I'll refer to y_h as the **homogeneous solution**.) If you think of $a_n D^n + \cdots + a_1 D + a_0$ as a machine which takes in functions and produces functions as its output, then feeding y_h to the machine gives 0:

$$y_h \quad \to \quad (a_n D^n + \cdots + a_1 D + a_0) \quad \to \quad 0$$

Undetermined coefficients will produce a **particular solution** y_p to the equation. When you feed y_p into the machine, you get $f(x)$:

$$y_p \quad \to \quad (a_n D^n + \cdots + a_1 D + a_0) \quad \to \quad f(x)$$

Because the equation is *linear*, if you feed $y_h + y_p$ into the machine, you get $0 + f(x) = f(x)$:

$$y_h + y_p \quad \to \quad (a_n D^n + \cdots + a_1 D + a_0) \quad \to \quad f(x)$$

$y_h + y_p$ is the general solution.

You may wonder why I have to use $y_h + y_p$ when y_p produces $f(x)$ by itself. The point is the y_h has the n arbitrary constants which make it a "general solution"; y_p doesn't have the constants, but it does produce $f(x)$. Together, they give a "general solution" which produces $f(x)$, which is what I want.

Examples.

$$y'' - 2y' + y = x^3 \cos 2x \quad \text{Undetermined coefficients applies.}$$

$$y'' - 2y' + y = \frac{1}{x} \quad \text{Undetermined coefficients doesn't apply.}$$

$$y'' - 2y' + y = 5x^7 e^{-3x} \quad \text{Undetermined coefficients applies.}$$

$$y'' - 2y' + y = \tan 2x \quad \text{Undetermined coefficients doesn't apply.} \ \Box$$

The problem comes down to finding a particular solution y_p. The idea is to guess the form of y_p, then substitute to find the coefficients. In order to guess y_p, I'll look at the *opposite* of the solution problem — given a function, find a differential equation which has the function as a solution.

Example. Find a D operator which gives the solution

$$y = c_1 e^{3x} + c_2 e^{-4x}.$$

The exponentials e^{3x} and e^{-4x} correspond to roots $m = 3$ and $m = -4$. Now $(m-3)(m+4) = m^2 + m - 12$, so

$$(D^2 + D - 12)y = 0$$

has $y = c_1 e^{3x} + c_2 e^{-4x}$ as its solution.

Another way of putting this is:

$$(D^2 + D - 12)(c_1 e^{3x} + c_2 e^{-4x}) = 0. \ \Box$$

Example. Find a D operator $f(D)$ such that

$$f(D)(2x + 7e^x) = 0.$$

The x corresponds to a root $m = 0$, but it must be a double root — m by itself gives rise to a constant solution c_1, but m^2 gives $c_2 x$. e^x comes from $m = 1$. Now $m^2(m-1) = m^3 - m^2$, so

$$(D^3 - D^2)(2x + 7e^x) = 0.$$

Check:

$$(D^3 - D^2)(2x + 7e^x) = D^3(2x + 7e^x) - D^2(2x + 7e^x) = (0 + 7e^x) - (0 + 7e^x) = 0.$$

Notice that the coefficients 2 and 7 were irrelevant. $\Box$

Example. Find a D operator $f(D)$ such that

$$f(D)(-e^x + e^{2x}\cos x - 3e^{2x}\sin x) = 0.$$

e^x corresponds to $m = 1$, which gives a factor of $m - 1$. $e^{2x}\cos x$ and $e^{2x}\sin x$ correspond to the conjugate complex roots $2 \pm i$. The corresponding polynomial is

$$(m - (2+i))(m - (2-i)) = m^2 - 4m + 5.$$

Therefore,

$$(D-1)(D^2-4m+5)(-e^x+e^{2x}\cos x-3e^{2x}\sin x)=0.\ \square$$

Example. Find a D operator $f(D)$ such that

$$f(D)(6x^2\sin 2x)=0.$$

$\sin 2x$ comes from $\pm 2i$, which comes from the factor

$$(m-2i)(m+2i)=m^2+4.$$

But the x^2 which multiplies the $\sin 2x$ indicates that this pair of roots is *tripled*. So the factor is $(m^2+4)^3$, and

$$(D^2+4)^3(6x^2\sin 2x)=0.\ \square$$

As I noted, I'll use this backward procedure to guess y_p. In the following examples, I'll put everything together and solve an equation using undetermined coefficients.

Example. $(D-3)^2y=e^x$.

Solve the homogeneous equation:

$$(D-3)^2y_h=0, \qquad y_h=c_1e^{3x}+c_2xe^{3x}.$$

Now I'll find y_p. Here's where I use the "backward procedure" of the previous examples. What D-thingie kills e^x? Clearly, $D-1$ does. Apply $D-1$ to both sides of the original equation:

$$D-3)^2y_p=e^x,$$

$$(D-1)(D-3)^2y_p=(D-1)e^x=0.$$

Therefore,

$$y_p=Ae^x+Be^{3x}+Cxe^{3x}.$$

(I'm using "A", "B", and "C" to avoid confusion with arbitrary constants.)

Throw out Be^{3x} and Cxe^{3x}, since these are homogeneous solutions. (There's no point feeding something into the original D-thingie which will give 0 — I'm looking for something which gives e^x.) What remains is my guess for y_p:

$$y_p=Ae^x.$$

The problem is to determine the value of A. (A is the *undetermined* coefficient.)

Compute the derivatives:

$$y_p=Ae^x, \quad y_p'=Ae^x, \quad y_p''=Ae^x.$$

Substitute in the original equation:

$$y_p''-6y_p'+9y_p=Ae^x-6Ae^x+9Ae^x=e^x,$$

$$4Ae^x=e^x.$$

Therefore, $4A=1$, so $A=\dfrac{1}{4}$. Hence,

$$y_p=\frac{1}{4}e^x.$$

The general solution is $y_h + y_p$:

$$y = c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{4} e^x.$$

You could have simply *guessed* $y_p = Ae^x$, reasoning that the only way to get an e^x out of the left side is to feed one in. However, the situation becomes more complicated if the right side happens to be a homogeneous solution. □

Example. $y'' - 4y' + 4y = e^{2x}$.

Find the homogeneous solution:

$$y_h'' - 4y_h' + 4y_h = 0,$$
$$(D^2 - 4D + 4)y_h = 0,$$
$$(D - 2)^2 y_h = 0,$$
$$y_h = c_1 e^{2x} + c_2 x e^{2x}.$$

Next, I have to find the particular solution. Note that it would not make sense to guess $y_p = Ae^{2x}$; since e^{2x} is a homogeneous solution, you *must* get 0 if you feed it into the equation!

The equation is

$$(D - 2)^2 y_p = e^{2x}.$$

The right side e^{2x} is a solution to $(D - 2)y = 0$, so apply $D - 2$ to both sides:

$$(D - 2)^3 y_p = 0.$$

Thus,

$$y_p = Ae^{2x} + Bxe^{2x} + Cx^2 e^{2x}.$$

Throw out Ae^{2x} and Bxe^{2x}, since they are homogeneous solutions. This leaves

$$y_p = Cx^2 e^{2x}.$$

Compute the derivatives:

$$y_p = Cx^2 e^{2x}, \quad y_p' = 2Cxe^{2x} + 2Cx^2 e^{2x}, \quad y_p'' = 2Ce^{2x} + 8Cxe^{2x} + 4Cx^2 e^{2x}.$$

Plug them into the original equation and solve for C:

$$\begin{array}{rclcrcr} y_p'' - 4y_p' + 4y_p & = & 2Ce^{2x} & + & 8Cxe^{2x} & + & 4Cx^2e^{2x} \\ & & & - & 8Cxe^{2x} & - & 8Cx^2e^{2x} \\ & & & & & + & 4Cx^2e^{2x} \\ & = & 2Ce^{2x} & = & e^{2x}. & & \end{array}$$

Therefore, $2C = 1$, so $C = \dfrac{1}{2}$. Hence,

$$y_p = \frac{1}{2} x^2 e^{2x}.$$

The general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^2 e^{2x}. \ \square$$

Example. $(D^2 - 1)y = e^x \sin x$.

Find the homogeneous solution:

$$(D-1)(D+1)y_h = 0,$$

$$y_h = c_1 e^x + c_2 e^{-x}.$$

Next, find the particular solution. When the right side contains expressions such as $e^x \sin x$, it's often easiest to use **complex exponentials**. In this case, notice that

$$e^x \sin x = \operatorname{im}\left(e^x(\cos x + i \sin x)\right) = \operatorname{im} e^x e^{ix} = \operatorname{im} e^{(1+i)x}.$$

Replace the original equation with the complexified equation

$$(D^2 - 1)y_p = e^{(1+i)x}.$$

Apply $D - (1+i)$ to both sides:

$$(D-1)(D+1)y_p = e^x \sin x,$$

$$(D-(1+i))\,(D-1)(D+1)y_p = 0,$$

$$y_p = Ae^{(1+i)x} + Be^x + Ce^{-x}.$$

Throw out the homogeneous solutions Be^x and Ce^{-x}, leaving

$$y_p = Ae^{(1+i)x}.$$

The derivatives are

$$y_p = Ae^{(1+i)x}, \quad y_p' = (1+i)Ae^{(1+i)x}, \quad y_p'' = (1+i)^2 Ae^{(1+i)x}.$$

Substitute:

$$y_p'' - y_p = (1+i)^2 Ae^{(1+i)x} - Ae^{(1+i)x} = \left((1+i)^2 - 1\right) Ae^{(1+i)x} = (-1+2i)Ae^{(1+i)x} = e^{(1+i)x}.$$

Therefore, $(-1+2i)A = 1$, so $A = -\dfrac{1}{5}(1+2i)$. The particular solution is

$$y_p = -\frac{1}{5}(1+2i)e^{(1+i)x}.$$

This is a complex solution. To get a particular solution for the original equation, I take the imaginary part of the solution above (since the original $e^x \sin x$ was the imaginary part of $e^{(1+i)x}$):

$$y_p = -\frac{1}{5}(1+2i)e^{(1+i)x} = -\frac{1}{5}(1+2i)e^x(\cos x + i \sin x) = -\frac{1}{5}e^x\left((\cos x - 2\sin x) + i(2\cos x + \sin x)\right),$$

$$\operatorname{im} y_p = -\frac{1}{5}e^x(2\cos x + \sin x).$$

(Likewise, if I'd had $e^x \cos x$ in the original equation, I'd have taken the *real* part.) $\operatorname{im} y_p$ is a particular solution for the original equation. The general solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{5}e^x(2\cos x + \sin x). \quad \square$$

Example. $(D^2 + 1)y = 4e^{-2x} \cos x$.

The homogeneous solution is

$$y_h = c_1 \cos x + c_2 \sin x.$$

To find a particular solution, complexify the equation:

$$(D^2 + 1)y_p = 4e^{(-2+i)x}.$$

Note that $e^{-2x} \cos x$ is the *real part* of $e^{(-2+i)x}$.
$e^{(-2+i)x}$ is killed by $D - (-2 + i)$, so apply $D - (-2 + i)$ to both sides:

$$(D - (-2 + i))(D^2 + 1)y_p = (D - (-2 + i))(4e^{(-2+i)x}) = 0.$$

Then

$$y_p = Ae^{(-2+i)x} + B \cos x + C \sin x.$$

$\cos x$ and $\sin x$ are homogeneous solutions, so throw them out. This leaves

$$y_p = Ae^{(-2+i)x}.$$

Differentiate:

$$y_p' = A(-2 + i)e^{(-2+i)x}, \qquad y_p'' = A(-2 + i)^2 e^{(-2+i)x}.$$

Substitute into the complexified equation:

$$y_p'' + y_p = A(-2 + i)^2 e^{(-2+i)x} + Ae^{(-2+i)x} = 4e^{(-2+i)x}.$$

Cancel the exponentials and simplify:

$$A\left[(-2 + i)^2 + 1\right] = 4, \quad A(4 - 4i) = 4, \quad A = \frac{1}{1 - i} = \frac{1}{2}(1 + i).$$

Therefore,

$$y_p = \frac{1}{2}(1 + i)e^{(-2+i)x}.$$

I have to expand this to recover the real part:

$$y_p = \frac{1}{2}(1 + i)e^{-2x}(\cos x + i \sin x) = \frac{1}{2}e^{-2x}\left[(\cos x - \sin x) + i(\cos x + \sin x)\right].$$

Hence, a particular solution to the *original* (uncomplexified) equation is given by

$$\operatorname{re} y_p = \frac{1}{2}e^{-2x}(\cos x - \sin x).$$

Therefore, the general solution is

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^{-2x}(\cos x - \sin x). \ \square$$

Example. Here are some examples where I'll just show what y_p to guess (as opposed to working out the coefficients).

For $y'' - 2y' + y = 3xe^x$,

$$y_h = c_1 e^x + c_2 xe^x, \quad \text{and you guess} \quad y_p = Ax^2e^x + Bx^3e^x.$$

For $y'' - 4y = 3x^2 - 7$,

$$y_h = c_1 e^{2x} + c_2 e^{-2x}, \quad \text{and you guess} \quad y_p = A + Bx + Cx^2.$$

For $(D^2 + 1)y = 3x \cos x$,

$$y_h = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

You guess

$$y_p = Ax^2 \cos x + Bx^2 \sin x + Cx^3 \cos x + Dx^3 \sin x.$$

Finally, for $(D^2 - D - 2)y = 5e^x \cos 3x$,

$$y_h = c_1 e^{2x} + c_2 e^{-x}.$$

You complexify the equation to

$$(D^2 - D - 2)y = 5e^{(1+3i)x}.$$

Then you guess

$$y_p = Ae^{(1+3i)x}.$$

After solving for A and finding y_p, you get a particular solution to the original equation by taking $\operatorname{re} y_p$.

□