1-18-1994

Direction fields and differential equations.

Name: ______________________________

This project is due by 3 p.m., Wednesday, February 2.

You will need the package `PlotField` to do this project. You should be able to load it using the following command:

```
Needs["Graphics`PlotField`"]
```

(Pay attention to the punctuation! The ` is a symbol called a **backquote**, a quote character which leans in the opposite direction from the normal single quote '. On my keyboard, it's located to the left of the `1` key.)

If *Mathematica* can't find the package, it may bring up a dialogue box and ask you to find it. It should be in the **Graphics** subdirectory (or folder) inside the **Packages** subdirectory (or folder) in the main *Mathematica* directory (or folder). On *MS-DOS* systems, the name of the file is **plotfiel.m**.

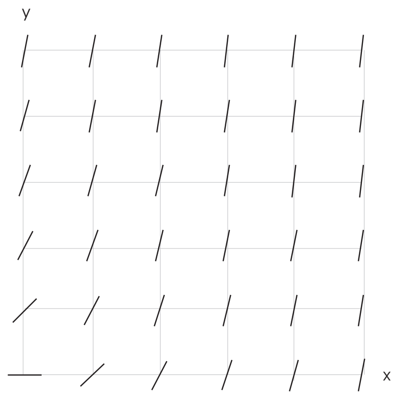
In this project, you will use **direction fields** to obtain qualitative information about solutions to differential equations. First, I'll review the construction of a direction field.

Here is a first-order differential equation:

$$y' = f(x, y).$$

If you plug a point (x, y) into the right side, you obtain y', the slope of the solution curve going through the point. The existence theorem for first-order equations says that, under reasonable conditions, there is a unique solution going through each point.

Set up a grid on the plane (i.e. use a piece of graph paper). At each point (x, y) on your grid, compute the slope y'. Draw a tiny segment with slope y' through the grid point. Do this for all the grid points. You'll get a picture like this:



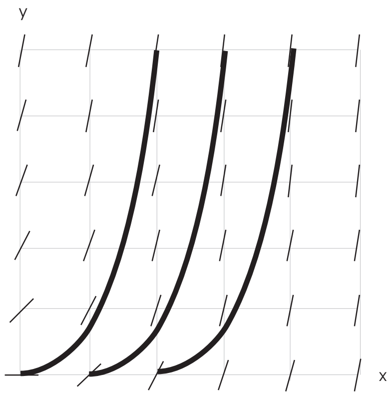
A picture of this sort is called a **direction field**. This picture shows the direction field for the equation

$$\frac{dy}{dx} = x + y.$$

For example, when $x = 1$ and $y = 1$ — i.e. at the point $(1, 1)$ — $\dfrac{dy}{dx} = 1 + 1 = 2$. So I drew a little segment with slope 2 through the point $(1, 1)$. Get out a ruler and line it up with the segment at $(1, 1)$. You can see that the edge of the ruler goes up 2 units (rise) for every 1 unit it goes to the right (run).

The little segments show the approximate shape of the solution curves. You get different solutions, as usual, by varying the "arbitary constant" C, or equivalently, by varying the initial condition.

Take a moment now to trace solution curves on the direction field above. Your curves don't have to pass through the little segments, but they should follow them in a rough way. Also, remember that solution curves *generally* do not cross (though crossings are possible at singular points).



You can do all of this by hand on a piece of graph paper, but it's easier to use *Mathematica*'s `PlotVectorField` function. First, I have to describe how to get a **vector field** from a differential equation. I'll use $\dfrac{dy}{dx} = 1 - y^2$ as an example.

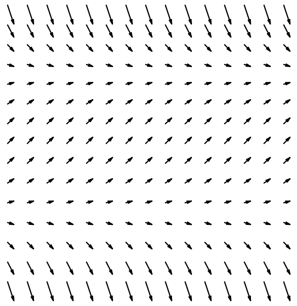
Recall from multivariable calculus that a vector field in the plane associates a vector $\vec{v} = \langle v_1, v_2 \rangle$ to each point in the plane. For example, the vector field might represent the velocity field of a fluid flow.

How do I get the two components of a vector from $\dfrac{dy}{dx} = 1 - y^2$? The idea is to think of $\dfrac{dy}{dx}$ as a quotient of a y-increment (dy) by an x-increment (dx). For example, I can set $dy = 1 - y^2$ and $dx = 1$, and then $\dfrac{dy}{dx} = \dfrac{1 - y^2}{1} = 1 - y^2$.

There isn't a unique way to represent $\dfrac{dy}{dx}$ as a quotient. I could write $dy = e^x(1 - y^2)$ and $dx = e^x$, and again $\dfrac{dy}{dx} = \dfrac{e^x(1 - y^2)}{e^x}$. There is usually an obvious way of doing it, and that will be fine for what I want to do.

Since I'm using $dx = 1$ and $dy = 1 - y^2$ as the components of my vector field, the field is $\vec{v} = \langle 1, 1 - y^2 \rangle$. Put it into the `PlotVectorField` function:

```
PlotVectorField[{1, 1 - y^2}, {x, -2, 2}, {y, -2, 2}]
```



(Be patient — *Mathematica* may take a while to draw the picture on slower computers.)

Notice that the arrows seem to be moving toward a horizontal line ($y = 1$) in the top half of the picture and moving away from a horizontal line ($y = -1$) in the bottom half of the picture.

The relative size of an arrow reflects the magnitude of $\frac{dy}{dx}$: Bigger arrows correspond to bigger rates of change (steep slopes), while smaller arrows correspond to smaller rates of change (gentle slopes).

A picture with lots of tiny arrows may be hard to interpret. Since I really only care about the slopes of the arrows, I can change the vector field so that all the arrows have length 1. Here's how.

Remember that you can use *Mathematica*'s online help to get the syntax of a command:

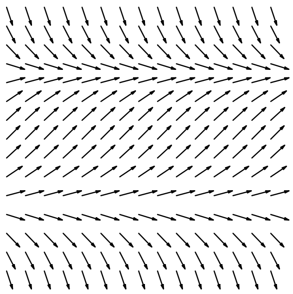
```
?PlotVectorField
```

If you want to see what options are available for this function, do:

```
Options[PlotVectorField]
```

The `ScaleFunction` option is the one I want. I will force *Mathematica* to use arrows of length 1:

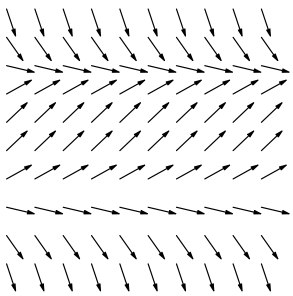
```
PlotVectorField[{1, 1 - y^2}, {x, -2, 2}, {y, -2, 2},
    ScaleFunction->(1 &)]
```



The new picture is a bit more legible than the old one. When you're doing the problems that follow, you may use scaling if you think it will give a nicer picture.

If the pictures take too long to draw on your computer, you can ask *Mathematica* to draw fewer arrows. The `PlotPoints` option gives the number of subdivisions in each direction. The default is 15; here's what happens if I change it to 10:

```
PlotVectorField[{1, 1 - y^2}, {x, -2, 2}, {y, -2, 2},
    ScaleFunction->(1 &), PlotPoints->10]
```



1. Consider the differential equation

$$y' = 4y - y^3.$$

(a) For what values of y does $\dfrac{dy}{dx} = 0$? (The corresponding solutions are called **equilibrium solutions**.)

(b) For what values of y is y increasing? For what values of y is y decreasing? (Recall, for example, that y increases where $\dfrac{dy}{dx} > 0$.)

(c) For what values of y do the solution curves to $y' = 4y - y^3$ have inflection points?

To do this, differentiate $y' = 4y - y^3$ *implicitly* with respect to x. Remember that, e.g., $\dfrac{d}{dx}(-y^3) = -3y^2y'$. This gives an equation $y'' = \ldots$. Substitute $y' = 4y - y^3$ into the right side to get everything in terms of y. Finally, set $y'' = 0$ and solve for y.

Next, you'll sketch the direction field to see if the solution curves exhibit the behavior predicted above.

2. (a) Use `PlotVectorField` to sketch the direction field differential equation $y' = 4y - y^3$.

(b) An equilibrium solution is **stable** if nearby solution curves are "attracted" to it; an equilibrium solution is **unstable** if nearby solution curves are "repelled" by it. In this example, what is the stable solution? What is the unstable solution?

3. Here is a **system of differential equations**:

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = -2x - y.$$

The solution would express x and y in terms of t. Since t does not appear on the right sides of the equations, you can sketch the direction field by writing

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2x - y}{-y}.$$

Use `PlotVectorField` to sketch the direction field.

4. (a) Use `PlotVectorField` to sketch the direction field for the differential equation

$$\frac{dy}{dx} = \frac{-x + y(1 - x^2 - y^2)}{y + x(1 - x^2 - y^2)}.$$

Use the ranges $-2 \le x \le 2$, $-2 \le y \le 2$. You will get a better picture if you use the option `ScaleFunction->(1 &)`.

(b) There is a solution which "attracts" the other solution curves. Label this solution on your picture. Look at the picture and at the differential equation. What is the *equation* of this attracting solution curve?

5. **(the pendulum equation)** Use `PlotVectorField` to sketch the direction field for the differential equation

$$\frac{dy}{dx} = \frac{-\sin x}{y}.$$

Use the ranges $0 \le x \le 10$, $-5 \le y \le 5$. In addition, sketch some of the solution curves. You will get a better picture if you use the option `ScaleFunction->(1 &)`.

The pendulum equation represents the motion of a (frictionless) pendulum, where x is the angle between the line of the pendulum and the vertical and y is the angular velocity. If you shove a pendulum to start it swinging, what happens after that depends on how hard you shoved.

A. If you give the pendulum a "moderate" shove, it simply swings back and forth.

B. If you give the pendulum a really hard shove, it will go through the top position and swing round and round in a circle.

(In between these cases, there is the situation where you shove the pendulum just hard enough to get it to stand straight up.)

6. Label the solution curves in the last picture you drew to show whether they represent case (A) or case (B). (Hint: In case (A), the pendulum's angular velocity (represented by the y-coordinate of a solution curve) must be 0 at the extreme points of the pendulum's swing. So in this case, you're looking for curves which cross the x-axis twice. In case (B), the angle between the pendulum and the vertical (i.e. the x-coordinate of a solution curve) must increase indefinitely as the pendulum goes round and round.)