Math 228
4-28-1994

Review Problems for the Final.

1. Solve the initial value problem

$$y'' + 3y' + 2y = 4t^2, \qquad y(0) = 0, \quad y'(0) = 0.$$

2. Solve for y in terms of x:

(a) $y'' + 4y' + 13y = 0$.

(b) $\left(D^2 + D\right)^2 y = 0$.

(c) $(D^2 - 5)y = 0$.

(d) $y'' - 2y' - 3y = 0, \qquad y(0) = 5, \quad y'(0) = 3$.

3. Solve the equation $(y^2 + 2xy)\,dx - x^2\,dy = 0$, $y(1) = 1$.

4. Solve: $y' = x - 2y\cot 2x$.

5. Solve: $y'' + y = \cot x$.

6. Solve the initial value problem

$$y'' - y = 4\delta(t-2) + t^2, \qquad y(0) = 0, \quad y'(0) = 2.$$

7. Find the Laplace transform of:

(a) $f(t) = 3\sin t + 4\cos 3t - t^2$

(b) $f(t) = e^t(t^2 + t + 1)$

(c) $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t^2 & \text{if } t \ge 2 \end{cases}$

(d) $f(t) = \begin{cases} t+1 & \text{if } 0 \le t < 3 \\ t+2 & \text{if } t \ge 3 \end{cases}$

(e) $f(t) = \begin{cases} \sin t & \text{if } 0 \le t < \dfrac{\pi}{4} \\ \sin t + \cos\left(t - \dfrac{\pi}{4}\right) & \text{if } t \ge \dfrac{\pi}{4} \end{cases}$

(f) $f(t) = \displaystyle\int_0^t e^{-(t-u)} \sin u\,du$

(g) $f(t) = u_3(t) \cdot (t^2 + 5)$

8. Solve:

(a) $(y^2 - 2xy + 6x)\,dx - (x^2 - 2xy + 2)\,dy = 0$, $y(1) = 1$.

(b) $(1 + y^2)\,dx + (x^2 y + y)\,dy = 0$, $y\left(\dfrac{\pi}{4}\right) = 0$.

9. (a) Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}.$$

(b) Find a matrix S such that $S^{-1}AS$ is diagonal.

(c) Solve the system

$$\vec{x}\,' = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \vec{x}.$$

10. Consider the linear system

$$\vec{x}\,' = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \vec{x},$$

where $\vec{x} = \langle x(t), y(t) \rangle$. Find the eigenvalues and solve the system directly. What happens to the solutions as $t \to \infty$? What does the phase portrait look like?

11. The equation

$$xy'' + 3y' - \frac{3}{x}y = x^2$$

has homogeneous solutions $y_1 = x$ and $y_2 = x^{-3}$. Find the general solution.

12. Find the following inverse transforms:

(a) $\mathcal{L}^{-1}\left(\dfrac{-3 - 8s - 7s^2 + 2s^3}{s^2(s+1)^2}\right)$.

(b) $\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2 + 2s + 2}\right)$.

(c) $\mathcal{L}^{-1}\left(\dfrac{e^{3s}}{(s-2)(s-4)}\right)$.

13. Find e^{tA}, if

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}.$$

14. Solve the system

$$\vec{x}\,' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \vec{x}.$$

Be sure to find two independent *real* solutions — there should be no complex numbers or functions in your answer!

Solutions to the Review Problems for the Final.

1. Solve the initial value problem

$$y'' + 3y' + 2y = 4t^2, \qquad y(0) = 0, \quad y'(0) = 0.$$

Take the Laplace transform of both sides and apply the initial value formula:

$$\mathcal{L}(y'') + 3\,\mathcal{L}(y') + 2\,\mathcal{L}(y) = \mathcal{L}(4t^2), \quad \left(-0 - 0 + s^2Y(s)\right) + 3\left(-0 + sY(s)\right) + 2Y(s) = \frac{8}{s^3}.$$

Solve for $Y(x)$:

$$Y(s) = \frac{8}{s^3(s+1)(s+2)}.$$

Apply partial fractions to the right side:

$$\frac{8}{s^3(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{s+2},$$

$$8 = As^2(s+1)(s+2) + Bs(s+1)(s+2) + C(s+1)(s+2) + Ds^3(s+2) + Es^3(s+1).$$

Setting $s = 0$ gives $8 = 2C$, so $C = 4$. Setting $s = -1$ gives $8 = -D$, so $D = -8$. Setting $s = -2$ gives $8 = 8E$, so $E = 1$.

Now

$$8 = As^2(s+1)(s+2) + Bs(s+1)(s+2) + 4(s+1)(s+2) - 8Ds^3(s+2) + s^3(s+1),$$

$$8 = A(s^4 + 3s^3 + 2s^2) + B(s^3 + 3s^2 + 2s) + 4(s^2 + 3s + 2) - 8(s^4 + 2s^3) + (s^4 + s^3).$$

Differentiate:

$$0 = A(4s^3 + 6s^2 + 4s) + B(3s^2 + 6s + 2) + 4(2s + 3) - 8(4s^3 + 6s^2) + (4s^3 + 3s^2).$$

Setting $s = 0$ gives $0 = 0 + 2B + 12 - 0 + 0$, so $B = -6$.

$$0 = A(4s^3 + 6s^2 + 4s) - 6(3s^2 + 6s + 2) + 4(2s + 3) - 8(4s^3 + 6s^2) + (4s^3 + 3s^2).$$

Differentiate:

$$0 = A(12s^2 + 12s + 4) - 6(6s + 6) + 8 - 8(12s^2 + 12s) + (12s^2 + 6s).$$

Setting $s = 0$ gives $0 = 4A - 36 + 8 - 0 + 0$, $4A = 28$, $A = 7$. Therefore,

$$Y(s) = \frac{7}{s} - \frac{6}{s^2} + \frac{4}{s^3} - \frac{8}{s+1} + \frac{1}{s+2}.$$

The solution is

$$y = 7 - 6t + 2t^2 - 8e^{-t} + e^{-2t}. \square$$

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2. Solve for y in terms of x:

(a) $y'' + 4y' + 13y = 0$.

The characteristic equation is $m^2 + 4m + 13 = 0$. The roots are $-2 \pm 3i$. The solution is

$$y = e^{-2x}\left(c_1 \cos 3x + c_2 \sin 3x.\right). \square$$

(b) $\left(D^2 + D\right)^2 y = 0$.

The characteristic equation is $m^2(m+1)^2 = 0$. The roots are 0 (double) and -1 (double). The solution is

$$y = c_1 + c_2 x + c_3 e^{-x} + c_4 x e^{-x}. \square$$

(c) $(D^2 - 5)y = 0$.

The characteristic equation is $m^2 - 5 = 0$. The roots are $\pm\sqrt{5}$. The solution is

$$y = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}. \square$$

(d) $y'' - 2y' - 3y = 0, \qquad y(0) = 5, \quad y'(0) = 3.$

The characteristic equation is $m^2 - 2m - 3 = 0$. The roots are 3 and -1. The solution is

$$y = c_1 e^{3x} + c_2 e^{-x}.$$

Plug in the initial condition:

$$5 = y(0) = c_1 + c_2.$$

Differentiate and plug in the other initial condition:

$$y' = 3c_1 e^{3x} - c_2 e^{-x}, \quad 3 = y'(0) = 3c_1 - c_2.$$

These equations yield $c_1 = 2$, $c_2 = 3$. The solution is

$$y = 2e^{3x} + 3e^{-x}. \square$$

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3. Solve the equation $(y^2 + 2xy)\,dx - x^2\,dy = 0$, $y(1) = 1$.

$$M_y = 2y + 2x, \quad N_x = -2x.$$

Hence, the equation is not exact. However,

$$\frac{N_x - M_y}{M} = \frac{-2y - 4x}{y^2 + 2xy} = \frac{-2(y + 2x)}{y(y + 2x)} = -\frac{2}{y}.$$

This is a function of y alone, so

$$\exp \int -\frac{2}{y}\,dy = \frac{1}{y^2}$$

is an integrating factor.

Multiply the equation by the integrating factor:

$$\left(1 + \frac{2x}{y}\right) dx - \frac{x^2}{y^2}\,dy = 0.$$

Now the equation is exact:

$$M_y = -\frac{2x}{y^2}, \quad N_x = -\frac{2x}{y^2}.$$

Integrate $M = 1 + \dfrac{2x}{y}$ with respect to x:

$$f = \int \left(1 + \frac{2x}{y}\right) dx = x + \frac{x^2}{y} + h(y).$$

Differentiate with respect to y;

$$-\frac{x^2}{y^2} = \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + \frac{dh}{dy}, \quad \frac{dh}{dy} = 0, \quad h = 0.$$

Hence,

$$f = x + \frac{x^2}{y} = C.$$

Plug in the initial condition:

$$1 + 1 = C, \quad C = 2.$$

The solution is

$$x + \frac{x^2}{y} = 2.$$

I dropped a solution in multiplying by $\dfrac{1}{y^2}$. There is no value of C for which the solution $x + \dfrac{x^2}{y} = C$ will yield $y = 0$. However, you can check by plugging in that $y = 0$ is a solution of the original equation (though not of the initial value problem). □

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4. Solve: $y' = x - 2y \cot 2x$.

Rewrite the equation as $y' + 2y \cot 2x = x$. The equation is first-order linear. We find an integrating factor:

$$I = \exp \int 2 \cot 2x \, dx = \exp(\ln \sin 2x) = \sin 2x.$$

Multiply by the integrating factor and solve for y:

$$y \sin 2x = \int x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C.$$

Here is the work for the integral:

$$\begin{array}{ccc} & \dfrac{d}{dx} & \int \, dx \\ + & x & \sin 2x \\ - & 1 & -\dfrac{1}{2} \cos 2x \\ + & 0 & -\dfrac{1}{4} \sin 2x \end{array}$$

$$\int x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C. \ \square$$

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5. Solve: $y'' + y = \cot x$.

I can't use undetermined coefficients, because $\cot x$ is not a solution to a linear constant coefficient homogeneous equation. Instead, I'll use variation of parameters. Since the equation is $(D^2 + 1)y = \cot x$, the homogeneous solutions are $y_1 = \cos x$ and $y_2 = \sin x$. The general solution will have the form

$$y = u_1 y_1 + u_2 y_2.$$

Set up the variation of parameters equation and solve for u_1 and u_2:

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \cot x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \cot x \end{bmatrix} = \begin{bmatrix} -\cos x \\ -\dfrac{(\cos x)^2}{\sin x} \end{bmatrix}.$$

Solve for u_1:

$$u_1' = -\cos x, \quad u_1 = -\sin x + c_1.$$

To solve for u_2, write

$$u_2' = -\frac{(\cos x)^2}{\sin x} = -\frac{1 - (\sin x)^2}{\sin x} = \csc x - \sin x.$$

Therefore,

$$u_2 = -\ln|\csc x + \cot x| + \cos x + c_2.$$

The solution is

$$y = -\sin x \cos x + c_1 \cos x - (\sin x)\ln|\csc x + \cot x| + \sin x \cos x + c_2 \sin x =$$

$$c_1 \cos x + c_2 \sin x - (\sin x)\ln|\csc x + \cot x|. \ \square$$

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6. Solve the initial value problem

$$y'' - y = 4\delta(t-2) + t^2, \qquad y(0) = 0, \quad y'(0) = 2.$$

Take the Laplace transform of both sides, applying the transform formula $\mathcal{L}(\delta(t-2)) = e^{-2s}$:

$$\left(-2 - 0 + s^2 Y(s)\right) - Y(s) = 4e^{-2s} + \frac{2}{s^3}, \quad (s^2 - 1)Y(s) = 2 + 4e^{-2s} + \frac{2}{s^3}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{2}{(s+1)(s-1)} + \frac{4}{(s+1)(s-1)}e^{-2s} + \frac{2}{s^3(s^2-1)}.$$

Now

$$\frac{2}{(s+1)(s-1)} = \frac{1}{s-1} - \frac{1}{s+1},$$

so

$$\mathcal{L}^{-1}\left(\frac{2}{(s+1)(s-1)}\right) = e^t - e^{-t},$$

$$\mathcal{L}^{-1}\left(\frac{4}{(s+1)(s-1)}e^{-2s}\right) = 2u_2(t)\left(e^{t-2} - e^{-(t-2)}\right).$$

For the remaining term, use the convolution formula:

$$\frac{2}{s^3(s^2-1)} = \frac{1}{s^3} \cdot \frac{2}{(s+1)(s-1)}.$$

Invert the factors:

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2}t^2, \quad \mathcal{L}^{-1}\left(\frac{2}{(s+1)(s-1)}\right) = e^t - e^{-t}.$$

Apply the convolution formula:

$$\mathcal{L}^{-1}\left(\frac{2}{s^3(s^2-1)}\right) = \int_0^t \frac{1}{2}u^2\left(e^{t-u} - e^{-(t-u)}\right)\,du =$$

$$\frac{1}{2}\left[e^{-t}\left(u^2e^u+2ue^u-2e^u\right)+e^t\left(-u^2e^{-u}-2ue^{-u}-2e^{-u}\right)\right]_0^t=-2+e^{-t}+e^t-t^2.$$

The solution is

$$y(t)=2e^t-2-t^2+2u_2(t)\left(e^{t-2}-e^{-(t-2)}\right).\ \square$$

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7. Find the Laplace transform of:

(a) $f(t)=3\sin t+4\cos 3t-t^2$

Using the transform table,

$$F(s)=\frac{3}{s^2+1}+\frac{4s}{s^2+9}-\frac{2}{s^3}.\ \square$$

(b) $f(t)=e^t(t^2+t+1)$

Apply the exponential shift:

$$F(s)=\frac{2}{(s-1)^3}+\frac{1}{(s-1)^2}+\frac{1}{s-1}.\ \square$$

(c) $f(t)=\begin{cases}0 & \text{if } 0\le t<2\\ t^2 & \text{if } t\ge 2\end{cases}$

$$t^2=(t-2)^2+4t-4=(t-2)^2+4(t-2)+4=g(t-2),$$

where $g(t)=t^2+4t+4$.

Since

$$\mathcal{L}\left(g(t)\right)=\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s},$$

it follows that

$$F(s)=e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right).\ \square$$

(d) $f(t)=\begin{cases}t+1 & \text{if } 0\le t<3\\ t+2 & \text{if } t\ge 3\end{cases}$

$$f(t)=(t+2)u_3(t)+(t+1)\left(1-u_1(t)\right)=t+1+u_3(t).$$

Therefore,

$$F(s)=\frac{1}{s^2}+\frac{1}{s}+\frac{1}{s}e^{-3s}.\ \square$$

(e) $f(t)=\begin{cases}\sin t & \text{if } 0\le t<\frac{\pi}{4}\\ \sin t+\cos\left(t-\frac{\pi}{4}\right) & \text{if } t\ge\frac{\pi}{4}\end{cases}$

$$f(t)=u_{\pi/4}(t)\left(\sin t+\cos\left(t-\frac{\pi}{4}\right)\right)+\left(1-u_{\pi/4}(t)\right)\sin t=\sin t+u_{\pi/4}(t)\cos\left(t-\frac{\pi}{4}\right).$$

Therefore,

$$F(s) = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}. \quad \square$$

(f) $f(t) = \int_0^t e^{-(t-u)} \sin u \, du$

$$f(t) = \int_0^t e^{-(t-u)} \sin u \, du = e^{-t} \star \sin t.$$

Therefore,

$$F(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+1}. \quad \square$$

(g) $f(t) = u_3(t) \cdot (t^2+5)$

$$t^2 + 5 = (t-3)^2 + 6(t-3) + 14 = g(t-3)$$

for $g(t) = t^2 + 6t + 14$.

Since

$$\mathcal{L}\left(t^2 + 6t + 14\right) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{14}{s},$$

it follows that

$$F(s) = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{14}{s}\right). \quad \square$$

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8. Solve:

(a) $(y^2 - 2xy + 6x)\,dx - (x^2 - 2xy + 2)\,dy = 0$, $y(1) = 1$.

$$M_y = 2y - 2x, \qquad N_x = -2x + 2y,$$

so the equation is exact.

Integrate M with respect to x:

$$f = \int (y^2 - 2xy + 6x)\,dx = xy^2 - x^2y + 3x^2 + h(y).$$

Differentiate with respect to y:

$$-x^2 + 2xy - 2 = \frac{df}{dy} = 2xy - x^2 + \frac{dh}{dy}, \quad \frac{dh}{dy} = -2, \quad h = 2y.$$

Therefore,

$$f = xy^2 - x^2y + 3x^2 + 2y = C.$$

Plug in the initial condition:

$$1 - 1 + 3 + 2 = C, \quad C = 5.$$

The solution is

$$xy^2 - x^2y + 3x^2 + 2y = 5. \quad \square$$

(b) $(1+y^2)\,dx + (x^2y+y)\,dy = 0$, $y\left(\frac{\pi}{4}\right) = 0$.

Separate variables:

$$\int \frac{1}{x^2+1}\,dx = -\int \frac{y}{y^2+1}, \quad \arctan x + C = -\frac{1}{2}\ln(y^2+1).$$

Plug in the initial condition:

$$1 + C = 0, \quad C = -1.$$

The solution is

$$\arctan x - 1 = -\frac{1}{2}\ln(y^2+1). \ \square$$

9. (a) Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 + \lambda - 2$; the eigenvalues are $\lambda = -2$ and $\lambda = 1$.
Consider $\lambda = -2$:

$$A + 2I = \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + b = 0$, or $a = -b$. The eigenvector is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ b \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Setting $b = 1$ yields $\langle -1, 1\rangle$.
For $\lambda = 1$,

$$A - I = \begin{bmatrix} -6 & -3 \\ 6 & 3 \end{bmatrix}$$

The rows are multiples. Ignore the first row, and divide the second row by 3 to get $\langle 2, 1\rangle$. I want a, b, such that

$$2 \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, -2\rangle$. $\square$

(b) Find a matrix S such that $S^{-1}\ A\ S$ is diagonal.

Construct S using the eigenvectors as its columns:

$$S = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Therefore,

$$S^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}.$$

You can check that

$$S^{-1}\ A\ S = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}. \ \square$$

(c) Solve the system

$$\vec{x}\,' = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \vec{x}.$$

The solution is

$$\vec{x} = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}. \ \square$$

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10. Consider the linear system

$$\vec{x}\,' = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \vec{x},$$

where $\vec{x} = \langle x(t), y(t) \rangle$. Find the eigenvalues and solve the system directly. What happens to the solutions as $t \to \infty$? What does the phase portrait look like?

The characteristic polynomial is $\lambda^2 + 4$; the eigenvalues are $\lambda = \pm 2i$. Therefore,

$$x = c_1 \cos 2t + c_2 \sin 2t.$$

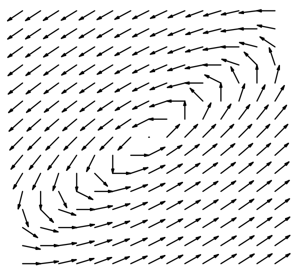
Now the first equation of the system says

$$y = -\frac{1}{4}(x' - 2x),$$

and $x' = -2c_1 \sin 2t + 2c_2 \cos 2t$, so

$$y = \left(\frac{1}{2}c_1 - \frac{1}{2}c_2\right) \cos 2t + \left(\frac{1}{2}c_1 + \frac{1}{2}c_2\right) \sin 2t.$$

Clearly, x and y oscillate as $t \to \infty$ (for any nonzero constants c_1, c_2). The only 2-dimensional phase portrait which produces this kind of behavior is a **center**.



This checks, since det $= 4$ while tr $= 0$. $\square$

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11. The equation

$$xy'' + 3y' - \frac{3}{x}y = x^2$$

has homogeneous solutions $y_1 = x$ and $y_2 = x^{-3}$. Find the general solution.

I'll use variation of parameters. First, divide through by x to put the equation in standard form:

$$y'' + \frac{3}{x}y' - \frac{3}{x^2}y = x.$$

Write down the variation of parameters equation:

$$\begin{bmatrix} x & x^{-3} \\ 1 & -3x^{-4} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = -\frac{1}{4}x^3 \begin{bmatrix} -3x^{-4} & -x^{-3} \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \frac{1}{4} \begin{bmatrix} x \\ -x^5 \end{bmatrix}.$$

Solve for u_1:

$$u_1' = \frac{1}{4}x, \quad u_1 = \frac{1}{8}x^2 + c_1.$$

Solve for u_2:

$$u_2' = -\frac{1}{4}x^5, \quad u_2 = -\frac{1}{24}x^6 + c_2.$$

The solution is

$$y = \frac{1}{8}x^3 + c_1 x - \frac{1}{24}x^3 + c_2 x^{-3} = c_1 x + c_2 x^{-3} + \frac{1}{12}x^3. \square$$

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12. Find the following inverse transforms:

(a) $\mathcal{L}^{-1}\left(\dfrac{-3 - 8s - 7s^2 + 2s^3}{s^2(s+1)^2}\right)$.

Compute the partial fractions decomposition:

$$\frac{-3 - 8s - 7s^2 + 2s^3}{s^2(s+1)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{(s+1)^2},$$

$$-3 - 8s - 7s^2 + 2s^3 = as(s+1)^2 + b(s+1)^2 + cs^2(s+1) + ds^2.$$

Setting $s = 0$ yields $b = -3$. Setting $s = 1$ yields $d = -4$.

$$-3 - 8s - 7s^2 + 2s^3 = as(s+1)^2 - 3(s+1)^2 + cs^2(s+1) - 4s^2.$$

Differentiate:

$$-8 - 14s + 6s^2 = a(3s^2 + 4s + 1) - 6(s+1) + c(3s^2 + 2s) - 8s.$$

Setting $s = 0$ yields $a = -2$.

$$-8 - 14s + 6s^2 = -2(3s^2 + 4s + 1) - 6(s+1) + c(3s^2 + 2s) - 8s.$$

Differentiate twice:

$$-14 + 12s = -2(6s + 4) - 6 + c(6s + 2) - 8, \quad 12 = -12 + 6c.$$

The last equation implies that $c = 4$. Therefore,

$$\frac{-3 - 8s - 7s^2 + 2s^3}{s^2(s+1)^2} = -\frac{2}{s} - \frac{3}{s^2} + \frac{4}{s+1} - \frac{4}{(s+1)^2}.$$

The inverse transform is

$$f(t) = -2 - 3t + 4e^{-t} - 4te^{-t}. \ \square$$

(b) $\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2+2s+2}\right)$.

Observe that

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}.$$

The exponential shift gives

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+2}\right) = e^{-t}\sin t.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2+2s+2}\right) = u_3(t)e^{-(t-3)}\sin(t-3). \ \square$$

(c) $\mathcal{L}^{-1}\left(\dfrac{e^{3s}}{(s-2)(s-4)}\right)$.

By partial fractions,

$$\frac{1}{(s-2)(s-4)} = \frac{1}{2}\left(\frac{1}{s-4} - \frac{1}{s-2}\right).$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{1}{(s-2)(s-4)}\right) = \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}.$$

Finally, apply the step function shift to obtain

$$\mathcal{L}^{-1}\left(\frac{e^{3s}}{(s-2)(s-4)}\right) = u_3(t)\left(\frac{1}{2}e^{4(t-3)} - \frac{1}{2}e^{2(t-3)}\right). \ \square$$

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13. Find e^{tA}, if

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 + 6\lambda + 9 = (\lambda+3)^2$. The eigenvalue is $\lambda = -3$ (double). Compute the a_k's:

$$a_1 = e^{-3t}, \quad a_2 = e^{-3t} \star a_1(t) = \int_0^t e^{-3(t-u)}e^{-3u}\,du = te^{-3t}.$$

Compute the B_k's:

$$B_1 = I, \quad B_2 = A + 3I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}.$$

Therefore,

$$e^{tA} = a_1 B_1 + a_2 B_2 = e^{-3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-3t} \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} e^{-3t} + 4te^{-3t} & -4te^{-3t} \\ 4te^{-3t} & e^{-3t} - 4te^{-3t} \end{bmatrix}. \quad \square$$

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14. Solve the system

$$\vec{x}\,' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \vec{x}.$$

Be sure to find two independent *real* solutions — there should be no complex numbers or functions in your answer!

The matrix

$$A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$$

has characteristic polynomial $\lambda^2 - 2\lambda + 10$. The eigenvalues are $\lambda = 1 \pm 3i$.

For $\lambda = 1 + 3i$,

$$A - (1+3i)I = \begin{bmatrix} 3 - 3i & -3 \\ 6 & -3 - 3i \end{bmatrix}$$

I want a nonzero vector $\langle a, b \rangle$ such that

$$\begin{bmatrix} 3 - 3i & -3 \\ 6 & -3 - 3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two rows of $A - (1+3i)I$ must be multiples. Otherwise, they'd be independent, and there wouldn't be any nonzero solutions. (This observation is always true for a 2×2 $A - \lambda I$ matrix that comes up in an eigenvector computation.)

So I can ignore the second row, and now I need to solve

$$\begin{bmatrix} 3 - 3i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This says

$$(3 - 3i)a + (-3)b = 0.$$

Obvious solution: Just swap the numbers $3 - 3i$ and -3 and negate one of them. Thus, $a = 3$ and $b = 3 - 3i$ works. This gives $\langle a, b \rangle = \langle a, 3 - 3i \rangle$.

The solution for this eigenvalue and eigenvector is

$$e^{(1+3i)t} \begin{bmatrix} 3 \\ 3 - 3i \end{bmatrix} = e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 3 \\ 3 - 3i \end{bmatrix} = e^t \begin{bmatrix} 3\cos 3t & + & 3i \sin 3t \\ (3\cos 3t + 3\sin 3t) & + & i(-3\cos 3t + 3\sin 3t) \end{bmatrix}.$$

Take the real and imaginary parts to obtain independent real solutions. The general solution is

$$\vec{x} = c_1 e^t \begin{bmatrix} 3\cos 3t \\ 3\cos 3t + 3\sin 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin 3t \\ -3\cos 3t + 3\sin 3t \end{bmatrix}. \quad \square$$

"The best thing for being sad is to learn something." - Merlyn, in T.H. White's *The Once and Future King*