Math 228
2-13-1994

Solutions to Problem Set 10.

[page 142]

7. Find the general solution of the differential equation $y'' - 2y' + 2y = 0$.

The characteristic equation is $m^2 - 2m + 2 = 0$. The roots are $m = 1 \pm i$. The solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x. \ \square$$

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12. Find the general solution of the differential equation $4y'' + 9y = 0$.

The characteristic equation is $4m^2 + 9 = 0$. The roots are $m = \pm \frac{3}{2} i$. The solution is

$$y = c_1 \cos \frac{3x}{2} + c_2 \sin \frac{3x}{2}. \ \square$$

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17. Find the solution of the initial value problem $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$. Sketch the graph of the solution and describe its behavior for increasing x.

The characteristic equation is $m^2 + 4 = 0$. The roots are $m = \pm 2i$. The general solution is

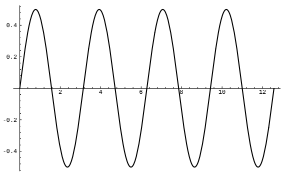
$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The derivative is $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$. Plug in the initial conditions:

$$0 = y(0) = c_1, \quad 1 = y'(0) = 2c_2.$$

It follows that $c_1 = 0$, $c_2 = \frac{1}{2}$. The solution is

$$y = \frac{1}{2} \sin 2x.$$



y oscillates as $x \to \infty$. $\square$

18. Find the solution of the initial value problem $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$. Sketch the graph of the solution and describe its behavior for increasing x.

The characteristic equation is $m^2 + 4m + 5 = 0$. The roots are $m = -2 \pm i$. The general solution is

$$y = e^{-2x}\left(c_1 \cos x + c_2 \sin x\right).$$

The derivative is

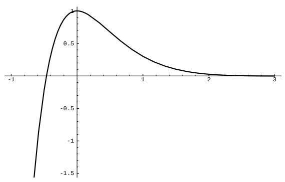
$$y' = e^{-2x}\left((-2c_1 + c_2)\cos x + (-c_1 - 2c_2)\sin x\right).$$

Plug in the initial conditions:

$$1 = y(0) = c_1, \quad 0 = y'(0) = -2c_1 + c_2.$$

It follows that $c_1 = 1$, $c_2 = 2$. The solution is

$$y = e^{-2x}\left(\cos x + 2\sin x\right).$$



As $x \to \infty$, $y \to 0$. $\square$

19. Find the solution of the initial value problem $y'' - 2y' + 5y = 0$, $y\left(\frac{\pi}{2}\right) = 0$, $y'\left(\frac{\pi}{2}\right) = 2$. Sketch the graph of the solution and describe its behavior for increasing x.

The characteristic equation is $m^2 - 2m + 5 = 0$. The roots are $m = 1 \pm 2i$. The general solution is

$$y = e^{x}\left(c_1 \cos 2x + c_2 \sin 2x\right).$$

Plug in the first initial condition:

$$0 = y\left(\frac{\pi}{2}\right) = e^{\pi/2}\left(c_1 \cdot (-1) + c_2 \cdot 0\right), \quad c_1 = 0.$$

Now $y = c_2 e^x \sin 2x$, so the derivative is

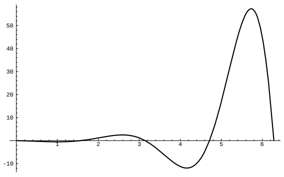
$$y' = c_2 e^x \sin 2x + 2c_2 e^x \cos 2x.$$

Plug in the second initial condition:

$$2 = y'\left(\frac{\pi}{2}\right) = c_2 e^{\pi/2} \cdot 0 + 2c_2 e^{\pi/2} \cdot (-1), \quad c_2 = -e^{-\pi/2}.$$

The solution is

$$y = -\exp\left(x - \frac{\pi}{2}\right)\sin 2x.$$



As $x \to \infty$, $y \to \pm\infty$. $\square$

I'll use the substitution $z = \ln x$ in the following problems. I might as well compute the derivatives once and for all. Since $x = e^z$,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x} = e^{-z}\frac{dy}{dz}.$$

Next,

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}\frac{d}{dz}\frac{dy}{dx} = e^{-z}\left(-e^{-z}\frac{dy}{dz} + e^{-z}\frac{d^2y}{dz^2}\right) = e^{-2z}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right).$$

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33. Use the result of Problem 32 to solve $x^2y'' + xy' + y = 0$.

Let $z = \ln x$, so $x = e^z$, $\dfrac{dy}{dx} = e^{-z}\dfrac{dy}{dz}$, and $\dfrac{d^2y}{dx^2} = e^{-2z}\left(-\dfrac{dy}{dz} + \dfrac{d^2y}{dz^2}\right)$.

$$e^{2z}\cdot e^{-2z}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right) + e^z\cdot e^{-z}\frac{dy}{dz} + y = 0.$$

Simplify:

$$\frac{d^2y}{dz^2} + y = 0.$$

The characteristic equation is $m^2 + 1 = 0$. The roots are $m = \pm i$. The solution is

$$y = c_1\cos z + c_2\sin z.$$

Put x back:

$$y = c_1\cos\ln x + c_2\sin\ln x.\ \square$$

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34. Use the result of Problem 32 to solve $x^2y'' + 4xy' + 2y = 0$.

Let $z = \ln x$, so $x = e^z$, $\dfrac{dy}{dx} = e^{-z}\dfrac{dy}{dz}$, and $\dfrac{d^2y}{dx^2} = e^{-2z}\left(-\dfrac{dy}{dz} + \dfrac{d^2y}{dz^2}\right)$.

$$e^{2z}\cdot e^{-2z}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right) + 4e^z\cdot e^{-z}\frac{dy}{dz} + 2y = 0.$$

Simplify:

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = 0.$$

The characteristic equation is $m^2 + 3m + 2 = 0$. The roots are $m = -1$, $m = -2$. The solution is

$$y = c_1 e^{-z} + c_2 e^{-2z}.$$

Put x back:

$$y = \frac{c_1}{x} + \frac{c_2}{x^2}. \square$$

"There are people who think that everything one does with a serious face is sensible." - G. C. Lichtenberg