Math 228
2-12-1994

Solutions to Problem Set 11.

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5. Find the Wronskian of $e^x \sin x$ and $e^x \cos x$.

$$W(e^x \sin x, e^x \cos x) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix} = -e^{2x}. \square$$

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9. Determine the largest interval in which the initial value problem $x(x-4)y'' + 3xy' + 4y = 2$, $y(3) = 0$, $y'(3) = -1$ is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

Rewrite the equation in the form

$$y'' + \frac{3}{x-4}y' + \frac{4}{x(x-4)}y = \frac{2}{x(x-4)}.$$

The functions $\frac{3}{x-4}$, $\frac{4}{x(x-4)}$, and $\frac{2}{x(x-4)}$ are continuous for $x \neq 0, 4$ — that is, on the intervals $x < 0$, $0 < x < 4$, and $x > 4$.

The initial conditions are given at $x = 3$, which is in the interval $0 < x < 4$. Hence, there is a unique solution to the initial value problem on the interval $0 < x < 4$. $\square$

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11. Determine the largest interval in which the initial value problem $(x-3)y'' + xy' + (\ln|x|)y = 0$, $y(1) = 0$, $y'(1) = 1$ is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

Rewrite the equation in the form

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0.$$

The functions $\frac{x}{x-3}$ and $\frac{\ln|x|}{x-3}$ are continuous for $x \neq 0, 3$ — that is, on the intervals $x < 0$, $0 < x < 3$, and $x > 3$.

The initial conditions were given at $x = 1$, which is in the interval $0 < x < 3$. Hence, there is a unique solution to the initial value problem on the interval $0 < x < 3$. $\square$

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14. Verify that $y_1(x) = 1$ and $y_2(x) = x^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $x > 0$. Then show that $c_1 + c_2 x^{1/2}$ is not, in general, a solution of this equation. Why not?

$y_1(x) = 1$ gives $y_1' = 0$ and $y_1'' = 0$. Clearly, $y_1 y_1'' + (y_1')^2 = 0$, so $y_1(x) = 1$ is a solution.
$y_2(x) = x^{1/2}$ gives $y_2' = \frac{1}{2}x^{-1/2}$ and $y_2'' = -\frac{1}{4}x^{-3/2}$. So

$$y_2 y_2'' + (y_2')^2 = -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0.$$

Therefore, $y_2(x) = x^{1/2}$ is a solution.

On the other hand, let $y = c_1 + c_2 x^{1/2}$. Then $y' = \dfrac{1}{2}c_2 x^{-1/2}$ and $y'' = -\dfrac{1}{4}c_2 x^{-3/2}$, so

$$yy'' + (y')^2 = -\frac{1}{4}c_1 c_2 x^{-3/2} - \frac{1}{4}c_2^2 x^{-1} + \frac{1}{4}c_2^2 x^{-1} = -\frac{1}{4}c_1 c_2 x^{-3/2}.$$

$\dfrac{1}{4}c_1 c_2 x^{-3/2} \neq 0$ unless $c_1 = 0$ or $c_2 = 0$, so $y = c_1 + c_2 x^{1/2}$ is not a solution, in general.

The point of the problem is that for linear equations a linear combination of solutions to be a solution. (That is, the sum of solutions is a solution, and a number times a solution is a solution.) This equation is nonlinear, so a linear combination of solutions need not be a solution. □

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4. Determine whether the functions e^{3x} and $e^{3(x-1)}$ are linearly independent or linearly dependent.

The Wronskian is

$$W\left(e^{3x}, e^{3(x-1)}\right) = \begin{vmatrix} e^{3x} & e^{3(x-1)} \\ 3e^{3x} & 3e^{3(x-1)} \end{vmatrix} = 3e^{6x-3} - 3e^{6x-3} = 0.$$

They are *not* independent.
In fact, it's easy to show this directly:

$$(-e^{-3}) \cdot e^{3x} + 1 \cdot e^{3(x-1)} = 0.$$

Since a nontrivial linear combination of the two functions is identically 0, they are dependent. □

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13. Find the Wronskian of two solutions of the differential equation $x^2 y'' - x(x+2)y' + (x+2)y = 0$ without solving the equation.

Write the equation in the form

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 0.$$

By Abel's theorem, the Wronskian is given by

$$W(y_1, y_2) = c\exp\left[-\int p(x)\,dx\right].$$

Since $p = -\dfrac{x+2}{x} = -1 - \dfrac{2}{x}$, we find that

$$W(y_1, y_2) = cx^2 e^x. \ \square$$

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23. Show that the function $f(x) = x|x|$ and $g(x) = x^2$ are linearly dependent on $0 < x < 1$ and on $-1 < x < 0$, but are linearly independent on $-1 < x < 1$. Although f and g are linearly independent there, show that $W(f, g)$ is zero for all x in $-1 < x < 1$. Hence f and g cannot be solutions of an equation $y'' + p(x)y' + q(x)y = 0$ with p and q continuous on $-1 < x < 1$.

First, I'll show that f and g are linearly dependent on $(0, 1)$. Since $x > 0$,

$$f(x) = x^2, \quad g(x) = x^2, \quad \text{so} \quad f(x) - g(x) = x^2 - x^2 = 0.$$

Next, I'll show that f and g are linearly dependent on $(-1, 0)$. Since $x < 0$,

$$f(x) = -x^2, \quad g(x) = x^2, \quad \text{so} \quad f(x) + g(x) = -x^2 + x^2 = 0.$$

Finally, I'll show that f and g are linearly independent on $(-1, 1)$. Suppose that there are numbers c_1 and c_2 such that

$$c_1 f(x) + c_2 g(x) = 0, \quad -1 < x < 1.$$

In particular, this must be true for $x = \frac{1}{2}$ and $x = -\frac{1}{2}$. Plugging these numbers in,

$$-\frac{1}{4}c_1 + \frac{1}{4}c_2 = 0, \quad \frac{1}{4}c_1 + \frac{1}{4}c_2 = 0.$$

Hence, $c_1 = 0$ and $c_2 = 0$. Therefore, f and g are independent on $-1 < x < 1$..

Nevertheless, the Wronskian vanishes for $-1 < x < 1$. If $x > 0$,

$$W(f, g) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0.$$

If $x < 0$,

$$W(f, g) = \begin{vmatrix} -x^2 & x^2 \\ -2x & 2x \end{vmatrix} = 0.$$

Finally, if $x = 0$,

$$W(f, g) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

Thus, $W(f, g) \equiv 0$ for $-1 < x < 1$.

This result shows that, in general, independence of functions is *not* equivalent to the Wronskian being nonzero. The point is that the functions f and g in this problem are *not* solutions to a linear second-order equation $y'' + p(x)y' + q(x)y = 0$, so there is no contradiction with the results of this section. $\square$

"Many of our cares are just morbid ways of looking at our privileges." - Walter Scott