Math 228
2-24-1994

Solutions to Problem Set 16.

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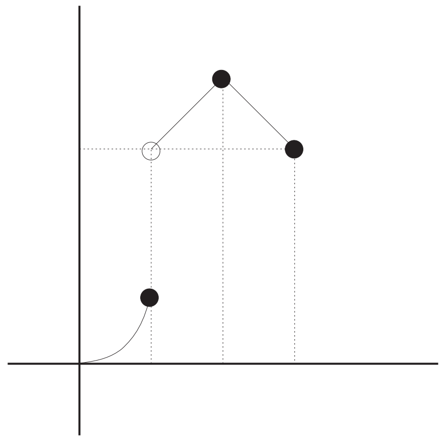
1. Sketch the graph of the function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t \le 1 \\ 2+t & \text{if } 1 < t \le 2 \\ 6-t & \text{if } 2 < t \le 3 \end{cases}$$

Determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

f is piecewise continuous.



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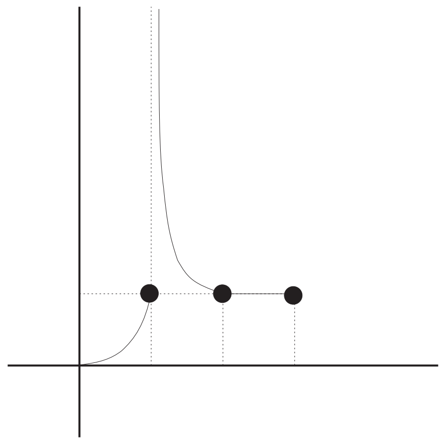
2. Sketch the graph of the function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t \le 1 \\ (t-1)^{-1} & \text{if } 1 < t \le 2 \\ 1 & \text{if } 2 < t \le 3 \end{cases}$$

Determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

f is not piecewise continuous.



7. Recall that $\cosh bt = \dfrac{e^{bt}+e^{-bt}}{2}$. Find the Laplace transform of $\cosh bt$; b is a real constant.

$$\mathcal{L}(\cosh bt) = \mathcal{L}\left(\frac{1}{2}\left(e^{bt}+e^{-bt}\right)\right) = \frac{1}{2}\left(\mathcal{L}\left(e^{bt}\right)+\mathcal{L}\left(e^{-bt}\right)\right) = \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right) = \frac{s}{s^2-b^2}. \quad \square$$

13. Recall that $\cos bt = \dfrac{e^{ibt}+e^{-ibt}}{2}$ and $\sin bt = \dfrac{e^{ibt}-e^{-ibt}}{2}$. Assuming that the necessary elementary integration formulas extend to this case, find the Laplace transform of $e^{at}\sin bt$; a and b are real constants.

Note that
$$e^{at}\sin bt = \operatorname{im}(e^{at}e^{bti}) = \operatorname{im} e^{(a+bi)t}.$$

I'll compute the transform of $e^{(a+bi)t}$, then take the imaginary part at the end.

$$\mathcal{L}\left(e^{(a+bi)t}\right) = \int_0^\infty e^{-st}e^{(a+bi)t}\,dt = \int_0^\infty e^{-(s-(a+bi))t}\,dt =$$
$$\lim_{p\to\infty}\left[-\frac{1}{(s-(a+bi))}e^{-(s-(a+bi))t}\right]_0^p = -\frac{1}{(s-(a+bi))}\lim_{p\to\infty}\left(e^{-(s-(a+bi))p}-1\right) =$$
$$\frac{1}{(s-(a+bi))} = \frac{1}{(s-a)-bi)}\frac{(s-a)+bi)}{(s-a)+bi)} = \frac{(s-a)+bi}{(s-a)^2+b^2}.$$

Therefore,
$$\mathcal{L}\left(e^{at}\sin bt\right) = \operatorname{im}\frac{(s-a)+bi}{(s-a)^2+b^2} = \frac{b}{(s-a)^2+b^2}. \quad \square$$

16. Using integration by parts, find the Laplace transform of $t\sin at$; a is a real constant.

$\sin at = \operatorname{im} e^{ati}$, so I will compute the transform of e^{ati}, then take the imaginary part at the end.

$$\mathcal{L}\left(te^{ati}\right) = \int_0^{+\infty} te^{-st}e^{ati}\,dt = \int_0^{+\infty} te^{(-s+ai)t}\,dt =$$
$$\left[\frac{t}{-s+ai}e^{(-s+ai)t} - \frac{1}{(-s+ai)^2}e^{(-s+ai)t}\right]_0^{+\infty} = \frac{1}{(-s+ai)^2} = \frac{1}{(s^2-a^2)-2ais} =$$
$$\frac{1}{(s^2-a^2)-2ais}\frac{(s^2-a^2)+2ais}{(s^2-a^2)+2ais} = \frac{(s^2-a^2)+2ais}{(s^2-a^2)^2+4a^2s^2} = \frac{(s^2-a^2)+2ais}{(s^2+a^2)^2}.$$

Therefore,
$$\mathcal{L}\left(t\sin at\right) = \operatorname{im}\frac{(s^2-a^2)+2ais}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}.$$

Here is the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dt} & & \int dt \\ + & t & & e^{(-s+ai)t} \\ & & \searrow & \\ - & 1 & & \dfrac{1}{-s+ai}e^{(-s+ai)t} \\ & & \searrow & \\ + & 0 & \rightarrow & \dfrac{1}{(-s+ai)^2}e^{(-s+ai)t} \end{array}$$

$$\int_0^{+\infty} te^{(-s+ai)t}\,dt = \frac{t}{-s+ai}e^{(-s+ai)t} - \frac{1}{(-s+ai)^2}e^{(-s+ai)t} + C. \square$$

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26. **The Gamma Function.** The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx.$$

This integral converges at infinity for all p. For $p < 0$ it is also improper because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show that for $p > 0$

$$\Gamma(p+1) = p\Gamma(p).$$

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\,dx.$$

Integrate the right side by parts. Here is the table:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x^p & & e^{-x} \\ & & \searrow & \\ - & px^{p-1} & \to & -e^{-x} \end{array}$$

Thus,

$$\int_0^\infty e^{-x}x^p\,dx = \left[-x^pe^{-x} + p\int e^{-x}x^{p-1}\,dx\right]_0^\infty = p\int_0^\infty e^{-x}x^{p-1}\,dx = p\Gamma(p). \square$$

(b) Show that $\Gamma(1) = 1$.

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 0 + 1 = 1. \square$$

(c) If p is a positive integer n, show that

$$\Gamma(n+1) = n!.$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

First, the result is true when $n = 0$ if $0!$ is defined to be 1, by part (b).

I'll prove the general result by induction. Suppose $n > 0$, and assume that $\Gamma(n) = (n-1)!$. I want to show $\Gamma(n+1) = n!$. By part (a),

$$\Gamma(n+1) = n\Gamma(n).$$

By the induction assumption, $\Gamma(n) = (n-1)!$, so

$$\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!.$$

This shows, by induction, that the formula is true for all $n \geq 0$. $\square$

"I have met people who found it astonishing that the cats have holes in their furs exactly at the places where the eyes are." - G. C. LICHTENBERG