Math 228
3-17-1994

Solutions to Problem Set 22.

1. Compute $\mathcal{L}^{-1}\left(\dfrac{1}{(s-1)(s^2+1)}\right)$.

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

By the convolution formula,

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)(s^2+1)}\right) = t \star e^t = \int_0^t e^{t-u} \sin u \, du = e^t \int_0^t e^{-u} \sin u \, du.$$

Integrate by parts:

$$\begin{array}{cccc} & \dfrac{d}{du} & & \int du \\ + & e^{-u} & & \sin u \\ & & \searrow & \\ - & -e^{-u} & & -\cos u \\ & & \searrow & \\ + & e^{-u} & \rightarrow & -\sin u \end{array}$$

$$\int e^{-u} \sin u \, du = -e^{-u} \cos u - e^{-u} \sin u - \int e^{-u} \sin u \, du,$$

$$\int e^{-u} \sin u \, du = -\frac{1}{2} e^{-u} (\cos u + \sin u) + C.$$

So

$$e^t \int_0^t e^{-u} \sin u \, du = e^t \left[-\frac{1}{2} e^{-u} (\cos u + \sin u)\right]_0^t = \frac{1}{2} e^t - \frac{1}{2} \cos t - \frac{1}{2} \sin t. \ \square$$

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2. Solve for y in terms of t:

$$y'' - 2y' + y = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 2t+2 & \text{if } t > 1 \end{cases}, \qquad y(0) = 0, y'(0) = -1.$$

First, I need to write the right side in terms of step functions:

$$t \cdot (u_0 - u_1) + (2t+2) \cdot u_1 = t u_0 + (t+2) u_1.$$

The transform of the first term is $\dfrac{1}{s^2}$. I match the second term against $f(t-c)u_c(t)$. $c = 1$, so $f(t-1) = t+2$. Therefore, $f(t) = t+3$, and

$$\mathcal{L}\left((t+2)u_1\right) = \left(\frac{1}{s^2} + \frac{3}{s}\right) e^{-s}.$$

Now go back and transform both sides of the differential equation:

$$\left(1 - 0 \cdot s + s^2 Y(s)\right) - 2\left(0 + sY(s)\right) + Y(s) = \frac{1}{s^2} + \left(\frac{1}{s^2} + \frac{3}{s}\right) e^{-s}.$$

Therefore,

$$Y(s) = -\frac{1}{(s-1)^2} + \frac{1}{s^2(s-1)^2} + \frac{1}{s^2(s-1)^2}e^{-s} + \frac{3}{s(s-1)^2}e^{-s}.$$

The first term $-\dfrac{1}{(s-1)^2}$ inverts to $-te^t$.

For the second and third terms, I need to invert $\dfrac{1}{s^2(s-1)^2}$. I'll use partial fractions:

$$\frac{1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2},$$

$$1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$$

Setting $s = 0$ gives $B = 1$. Setting $s = 1$ gives $D = 1$. The equation now reads

$$1 = As(s-1)^2 + (s-1)^2 + Cs^2(s-1) + s^2.$$

Differentiate:

$$0 = A(s-1)^2 + 2As(s-1) + 2(s-1) + 2Cs(s-1) + Cs^2 + 2s.$$

Setting $s = 0$ gives $A = 2$. Setting $s = 1$ gives $C = -2$. Thus,

$$\frac{1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^2}\right) = 2 + t - 2e^t + te^t = g(t).$$

(I'm naming the expression "$g(t)$" for convenience.)

$g(t)$ is the inverse of the second term in $Y(s)$. By the step function transform formula, the inverse of $\dfrac{1}{s^2(s-1)^2}e^{-s}$ is simply $g(t-1)u_1(t)$.

Finally, I'll invert the fourth term in $Y(s)$. By the step function transform formula, I first need to invert $\dfrac{3}{s(s-1)^2}$. I'll use the convolution formula. First,

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = te^t.$$

So

$$\mathcal{L}^{-1}\left(\frac{3}{s(s-1)^2}\right) = 3(1 \star te^t) = 3\int_0^t ue^u\,du = 3\left[ue^u - e^u\right]_0^t = 3(te^t - e^t + 1) = h(t).$$

(I'm naming the expression "$h(t)$" for convenience.)

Therefore,

$$\mathcal{L}^{-1}\left(\frac{3}{s(s-1)^2}e^{-s}\right) = h(t-1)u_1(t).$$

Hence,

$$y(t) = -te^t + g(t) + g(t-1)u_1(t) + h(t-1)u_1(t). \quad \square$$

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3. Solve for y in terms of t:

$$y'' + 3y' + 2y = e^{-3t}u_2(t), \qquad y(0) = 2, y'(0) = -3.$$

I'm going to take the transform of both sides; I'll do the transform of the right side first. Match the right side with the step function transform formula: I get $f(t-c)u_c(t) = e^{-3t}u_2(t)$, so $c = 2$ and $f(t-2) = e^{-3t}$. Then $f(t) = e^{-3(t+2)} = e^{-6}e^{-3t}$. Hence, $F(s) = \mathcal{L}[f(t)] = e^{-6}\dfrac{1}{s+3}$. Finally,

$$\mathcal{L}\left(e^{-3t}u_2(t)\right) = e^{-6}\frac{1}{s+3}e^{-2s}.$$

Now go back to the original equation and transform both sides:

$$\left(3 - 2s + s^2 Y(s)\right) + 3\left(-2 + sY(s)\right) + 2Y(s) = e^{-6}\frac{1}{s+3}e^{-2s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{2s+3}{(s+1)(s+2)} + e^{-6}\frac{1}{(s+1)(s+2)(s+3)}e^{-2s}.$$

The first term can be decomposed by inspection:

$$\frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{2s+3}{(s+1)(s+2)}\right) = e^{-t} + e^{-2t}.$$

The second term requires a step function shift. I need to invert $\dfrac{1}{(s+1)(s+2)(s+3)}$. The easiest way to do this is to use partial fractions; I'll omit the ugly but routine algebra and simply note that

$$\frac{1}{(s+1)(s+2)(s+3)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+2)(s+3)}\right) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t} = g(t).$$

Therefore, by the step function transform formula,

$$\mathcal{L}^{-1}\left(e^{-6}\frac{1}{(s+1)(s+2)(s+3)}e^{-2s}\right) = e^{-6}g(t-2)u_2(t).$$

The solution is

$$y(t) = e^{-t} + e^{-2t} + e^{-6}g(t-2)u_2(t). \;\square$$

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4. Use Laplace transforms to compute $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+1}\,dx$.

Define

$$f(t) = \int_0^\infty \frac{x\sin(xt)}{x^2+1}\,dx.$$

The original integral is $f(1)$.
To compute $f(t)$, I'll take the Laplace transform:

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^\infty \frac{x\sin(xt)}{x^2+1}\,dx\right] = \int_0^\infty \frac{x}{x^2+1}\mathcal{L}(\sin xt)\,dx = \int_0^\infty \frac{x}{x^2+1}\frac{x}{x^2+s^2}\,dx.$$

By partial fractions,

$$\frac{x^2}{(x^2+1)(x^2+s^2)} = \frac{1}{1-s^2}\left(\frac{1}{x^2+1} - \frac{s^2}{x^2+s^2}\right).$$

Therefore,

$$F(s) = \frac{1}{1-s^2}\int_0^\infty \left(\frac{1}{x^2+1} - \frac{s^2}{x^2+s^2}\right) dx = \frac{1}{1-s^2}\left[\tan^{-1} x - s\tan^{-1}\frac{x}{s}\right]_0^\infty = \frac{1}{1-s^2}\cdot\frac{\pi}{2}(1-s) = \frac{\pi}{2}\frac{1}{s+1}.$$

Invert the transform to find f:

$$f(t) = \frac{\pi}{2}e^{-t}.$$

Hence,

$$\int_0^\infty \frac{x\sin x}{x^2+1}\,dx = f(1) = \frac{\pi}{2e}. \square$$

"Most people would rather die than think. In fact, they do." - Bertrand Russell