Math 228
4-2-1994

Solutions to Problem Set 27.

1. Compute e^{tA}, if:

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Since A is a diagonal matrix — all the non-diagonal entries are 0 — I just "exponentiate" the diagonal entries:

$$e^{tA} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}. \quad \square$$

(b) $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

First, compute some powers of A:

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ -9 & 1 \end{bmatrix}.$$

I can see that the pattern is

$$A^n = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ -3n & 1 \end{bmatrix}.$$

Therefore,

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \dfrac{t^n}{n!} & 0 \\ \sum_{n=0}^{\infty} \dfrac{-3nt^n}{n!} & \sum_{n=0}^{\infty} \dfrac{t^n}{n!} \end{bmatrix}.$$

First,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

For the other sum,

$$\sum_{n=0}^{\infty} \frac{-3nt^n}{n!} = -3 \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} = -3t \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = -3t \sum_{m=0}^{\infty} \frac{t^m}{m!} = -3te^t.$$

(Let $m = n - 1$. When $n = 1$, $m = 0$; as $n \to \infty$, $m \to \infty$.)
Therefore,

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3te^t & e^t \end{bmatrix}. \quad \square$$

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2. Use diagonalization to compute e^{tA}, if:

(a) $A = \begin{bmatrix} 1 & 0 \\ -6 & 5 \end{bmatrix}$

The characteristic polynomial is $5 - 6\lambda + \lambda^2$; the eigenvalues are $\lambda = 5$ and $\lambda = 1$.
Consider the eigenvalue $\lambda = 5$:

$$A - 5I = \begin{bmatrix} -4 & 0 \\ -6 & 0 \end{bmatrix} \underset{r_1 \to -r_1/4}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ -6 & 0 \end{bmatrix} \underset{r_2 \to r_2 + 6r_1}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a = 0$, so the eigenvectors are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Setting $b = 1$ produces the eigenvector $\langle 0, 1 \rangle$.
Next, consider the eigenvalue $\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 0 \\ -6 & 4 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\longrightarrow} \begin{bmatrix} -6 & 4 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to -r_1/6}{\longrightarrow} \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \frac{2}{3}b = 0$, or $a = \frac{2}{3}b$. The eigenvectors are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{2}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Setting $b = 3$ gives $\langle 2, 3 \rangle$.
The matrix whose columns are the eigenvectors is

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Its inverse is

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}.$$

Therefore,

$$e^{tA} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ \frac{3}{2}e^t - \frac{3}{2}e^{5t} & e^{5t} \end{bmatrix}. \quad \square$$

(b) $A = \begin{bmatrix} -4 & 4 \\ -6 & 6 \end{bmatrix}$

The characteristic polynomial is $-2\lambda + \lambda^2$; the eigenvalues are $\lambda = 0$ and $\lambda = 2$.

Consider the eigenvalue $\lambda = 0$:

$$A - 0 \cdot I = \begin{bmatrix} -4 & 4 \\ -6 & 6 \end{bmatrix} \underset{r_1 \to -r_1/4}{\longrightarrow} \begin{bmatrix} 1 & -1 \\ -6 & 6 \end{bmatrix} \underset{r_2 \to r_2 + 6r_1}{\longrightarrow} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - b = 0$, or $a = b$. The eigenvectors are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Setting $b = 1$ produces the eigenvector $\langle 1, 1 \rangle$.
Now consider the eigenvalue $\lambda = 2$:

$$A - 2I = \begin{bmatrix} -6 & 4 \\ -6 & 4 \end{bmatrix} \underset{r_1 \to -r_1/6}{\longrightarrow} \begin{bmatrix} 1 & -\frac{2}{3} \\ -6 & 4 \end{bmatrix} \underset{r_2 \to r_2 + 6r_1}{\longrightarrow} \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \frac{2}{3}b = 0$, or $a = \frac{2}{3}b$. The eigenvectors are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{2}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Taking $b = 3$ yields $\langle 2, 3 \rangle$.
The matrix whose columns are the eigenvectors is

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Its inverse is

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$e^{tA} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 - 2e^{2t} & -2 + 2e^{2t} \\ 3 - 3e^{2t} & -2 + 3e^{2t} \end{bmatrix}. \quad \square$$

(c) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

In this problem, the eigenvalues and eigenvectors were given . The matrix whose columns are the eigenvectors is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Its inverse is

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Therefore,

$$e^{tA} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \frac{1}{2}e^t + \frac{1}{2}e^{2t} & \frac{1}{2}e^t - \frac{1}{2}e^{2t} \\ 0 & \frac{1}{2}e^t - \frac{1}{2}e^{2t} & \frac{1}{2}e^t + \frac{1}{2}e^{2t} \end{bmatrix}. \quad \square$$

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3. Show that there is no matrix A such that

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3t & e^t \end{bmatrix}.$$

Observe that

$$\left.\frac{d}{dt}e^{tA}\right|_{t=0} = \left(Ae^{tA}\right)\big|_{t=0} == A \cdot I = A.$$

Now suppose that that was a matrix A such that

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3t & e^t \end{bmatrix}.$$

Differentiate both sides and set $t = 0$:

$$A = \left.\begin{bmatrix} e^t & 0 \\ -3 & e^t \end{bmatrix}\right|_{t=0} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

However, by Problem 1(b),

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ -3te^t & e^t \end{bmatrix},$$

and

$$\begin{bmatrix} e^t & 0 \\ -3te^t & e^t \end{bmatrix} \neq \begin{bmatrix} e^t & 0 \\ -3t & e^t \end{bmatrix}.$$

This contradiction shows that there is no such matrix A. $\square$

"Seek simplicity and distrust it." - Alfred North Whitehead