Math 228
4-10-1994

Solutions to Problem Set 29.

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I'll solve the following systems by computing the matrix exponential e^{At}. Then the solution is given by

$$\vec{x} = e^{At} \int e^{-At}\vec{b}\,dt + e^{At}\vec{c}.$$

1. Find the general solution of the system of equations:

$$\vec{x}\,' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 - 1$. The eigenvalues are $\lambda = 1$ and $\lambda = -1$.
First, I'll compute the a_k's. $a_1 = e^t$, and

$$a_2 = e^{-t} \star e^t = \int_0^t e^{-(t-u)} e^u\,du = e^{-t} \int_0^t e^{2u}\,du = \frac{1}{2}e^t - \frac{1}{2}e^{-t}.$$

Next, I'll compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}.$$

Therefore,

$$e^{At} = a_1 B_1 + a_2 B_2 = \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}.$$

I'll do the rest of the computation in steps. First,

$$e^{-At}\vec{b} = \frac{1}{2} \begin{bmatrix} 3e^{-t} - e^t & -e^{-t} + e^t \\ 3e^{-t} - 3e^t & -e^{-t} + 3e^t \end{bmatrix} \begin{bmatrix} e^t \\ t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 - 2e^{2t} - te^{-t} + te^t \\ 3 - 3e^{2t} - te^{-t} + 3te^t \end{bmatrix}.$$

Next,

$$\int e^{-At}\vec{b}\,dt = \frac{1}{2} \begin{bmatrix} 3t - \dfrac{1}{2}e^{2t} + te^{-t} + e^{-t} + te^t - e^t \\ 3t - \dfrac{3}{2}e^{2t} + te^{-t} + e^{-t} + 3te^t - 3e^t \end{bmatrix}.$$

The solution is

$$\vec{x} = \frac{1}{4} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 3t - \dfrac{1}{2}e^{2t} + te^{-t} + e^{-t} + te^t - e^t \\ 3t - \dfrac{3}{2}e^{2t} + te^{-t} + e^{-t} + 3te^t - 3e^t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix} \vec{c} =$$

$$\begin{bmatrix} -\dfrac{1}{4}e^t + t + \dfrac{3}{2}te^t \\ -\dfrac{3}{4}e^t + 2t + \dfrac{3}{2}te^t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix} \vec{c}. \square$$

7. Find the general solution of the system of equations:

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t.$$

The characteristic polynomial is $\lambda^2 - 2\lambda - 3$. The eigenvalues are $\lambda = 3$ and $\lambda = -1$. First, I'll compute the a_k's. $a_1 = e^{3t}$, and

$$a_2 = e^{-t} \star e^{3t} = \int_0^t e^{-(t-u)} e^{3y}\, du = e^{-t} \int e^{4u}\, du = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}.$$

Next, I'll compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}.$$

Therefore,

$$e^{At} = a_1 B_1 + a_2 B_2 = \frac{1}{4} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix}.$$

I'll do the rest of the computation in steps. First,

$$e^{-At}\vec{b} = \frac{1}{4} \begin{bmatrix} 2e^{-3t} + 2e^t & e^{-3t} - e^t \\ 4e^{-3t} - 4e^t & 2e^{-3t} + 2e^t \end{bmatrix} \begin{bmatrix} 2e^t \\ -e^t \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3e^{-2t} + 5e^{2t} \\ 6e^{-2t} - 10e^{2t} \end{bmatrix}.$$

Next,

$$\int e^{-At}\vec{b}\, dt = \frac{1}{8} \begin{bmatrix} -3e^{-2t} + 5e^{2t} \\ -6e^{-2t} - 10e^{2t} \end{bmatrix}.$$

The solution is

$$\vec{x} = \frac{1}{32} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix} \begin{bmatrix} -3e^{-2t} + 5e^{2t} \\ -6e^{-2t} - 10e^{2t} \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix} \vec{c} =$$

$$\frac{1}{4} \begin{bmatrix} e^t \\ -8e^t \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{bmatrix} \vec{c}. \quad \square$$

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3. Find approximate values of the solution of the initial value problem $y' = t^2 + y^2$, $y(0) = 1$, at $t = t_0 + 0.1$, $t_0 + 0.2$, $t_0 + 0.3$, and $t_0 + 0.4$:

Here is a *Mathematica* function which will do the computations.

```
euler[f_, xstart_, ystart_, h_, num_] :=
Block[{x = xstart, y = ystart, i = 0, mytable={}},
    While[i <= num,
        AppendTo[mytable, {x, y, f[x,y], f[x,y] h}];
        x = x + h;
        y = y + f[x,y] h;
        i = i + 1;
    ];
    mytable]
```

For example, for Problem 3(a), define the function:

```
f3[t_, y_] := t^2 + y^2
```

Then make the table:

```
euler[f3, 0.0, 1.0, 0.1, 0.4/0.1]
```

That is, use Euler's method on `f3` with $t_0 = 0.0$ and $y_0 = 1.0$. Use an increment $h = 0.1$, and take $0.4/0.1 = 4$ steps.

(a) Using the Euler method with $h = 0.1$.

t	y	$\frac{dy}{dt}$	dy
0.	1.	1.	0.1
0.1	1.1	1.22	0.122
0.2	1.222	1.53328	0.153328
0.3	1.37533	1.98153	0.198153
0.4	1.57348	2.63584	0.263584 $\square$

(b) Using the Euler method with $h = 0.05$

t	y	$\frac{dy}{dt}$	dy
0.	1.	1.	0.05
0.05	1.05	1.105	0.05525
0.1	1.10525	1.23158	0.0615789
0.15	1.16683	1.38399	0.0691995
0.2	1.23603	1.56777	0.0783883
0.25	1.31442	1.79019	0.0895096
0.3	1.40393	2.06101	0.10305
0.35	1.50698	2.39348	0.119674
0.4	1.62665	2.80599	0.1403 $\square$

(c) Using the Euler method with $h = 0.025$

t	y	$\frac{dy}{dt}$	dy
0.	1.	1.	0.025

0.025	1.02502	1.05128	0.0262821
0.05	1.05134	1.10783	0.0276956
0.075	1.07912	1.17012	0.029253
0.1	1.10848	1.23873	0.0309682
0.125	1.13959	1.31429	0.0328572
0.15	1.17262	1.39753	0.0349384
0.175	1.20776	1.48931	0.0372327
0.2	1.24523	1.59059	0.0397648
0.225	1.28526	1.70251	0.0425628
0.25	1.32812	1.8264	0.0456599
0.275	1.37411	1.96379	0.0490948
0.3	1.42356	2.11652	0.052913
0.325	1.47686	2.28675	0.0571687
0.35	1.53445	2.47705	0.0619262
0.375	1.59683	2.6905	0.0672625
0.4	1.66458	2.93083	0.0732707

□

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9. Find an approximate value of the solution of the initial value problem $y' = 0.5 - t + 2y$, $y(0) = 1$ at $t = t_0 + 1$:

(a) Using the Euler method with $h = 0.025$.

t	y	$\frac{dy}{dt}$	dy
0.	1.	2.5	0.0625
0.025	1.06187	2.59875	0.0649688
0.05	1.12622	2.70244	0.0675609
0.075	1.19315	2.81131	0.0702827
0.1	1.26281	2.92562	0.0731406
0.125	1.33533	3.04566	0.0761414

0.15	1.41084	3.17169	0.0792922
0.175	1.48951	3.30402	0.0826006
0.2	1.57149	3.44297	0.0860744
0.225	1.65694	3.58887	0.0897218
0.25	1.74603	3.74207	0.0935517
0.275	1.83896	3.90292	0.097573
0.3	1.93591	4.07182	0.101795
0.325	2.03708	4.24916	0.106229
0.35	2.14268	4.43536	0.110884
0.375	2.25294	4.63088	0.115772
0.4	2.36809	4.83618	0.120904
0.425	2.48837	5.05174	0.126293
0.45	2.61404	5.27807	0.131952
0.475	2.74536	5.51573	0.137893
0.5	2.88263	5.76526	0.144132
0.525	3.02614	6.02728	0.150682
0.55	3.17619	6.30239	0.15756
0.575	3.33313	6.59126	0.164781
0.6	3.49729	6.89457	0.172364
0.625	3.66903	7.21305	0.180326
0.65	3.84873	7.54745	0.188686
0.675	4.03679	7.89858	0.197464
0.7	4.23363	8.26726	0.206681
0.725	4.43968	8.65437	0.216359
0.75	4.65542	9.06084	0.226521
0.775	4.88131	9.48763	0.237191
0.8	5.11788	9.93576	0.248394

0.825	5.36565	10.4063	0.260157
0.85	5.62518	10.9004	0.272509
0.875	5.89707	11.4191	0.285478
0.9	6.18192	11.9638	0.299096
0.925	6.48039	12.5358	0.313394
0.95	6.79316	13.1363	0.328408
0.975	7.12094	13.7669	0.344172
1.	7.46449	14.429	0.360724

(b) Using the Euler method with $h = 0.0125$.

t	y	$\frac{dy}{dt}$	dy
0.	1.	2.5	0.03125
0.0125	1.03109	2.54969	0.0318711
0.025	1.06281	2.60062	0.0325077
0.0375	1.09516	2.65282	0.0331603
0.05	1.12816	2.70633	0.0338291
0.0625	1.16184	2.76117	0.0345147
0.075	1.1962	2.81739	0.0352174
0.0875	1.23126	2.87501	0.0359377
0.1	1.26704	2.93408	0.0366759
0.1125	1.30356	2.99462	0.0374327
0.125	1.34083	3.05667	0.0382084
0.1375	1.37889	3.12027	0.0390034
0.15	1.41773	3.18547	0.0398183
0.1625	1.4574	3.25229	0.0406536
0.175	1.49789	3.32079	0.0415098
0.1875	1.53925	3.39099	0.0423874

0.2	1.58148	3.46295	0.0432869
0.2125	1.62461	3.53672	0.044209
0.225	1.66866	3.61232	0.045154
0.2375	1.71366	3.68982	0.0461227
0.25	1.75963	3.76925	0.0471156
0.2625	1.80658	3.85067	0.0481334
0.275	1.85456	3.93412	0.0491765
0.2875	1.90358	4.01966	0.0502458
0.3	1.95367	4.10734	0.0513418
0.3125	2.00486	4.19721	0.0524652
0.325	2.05717	4.28933	0.0536166
0.3375	2.11063	4.38375	0.0547969
0.35	2.16527	4.48053	0.0560067
0.3625	2.22112	4.57973	0.0572467
0.375	2.27821	4.68142	0.0585177
0.3875	2.33657	4.78564	0.0598205
0.4	2.39623	4.89247	0.0611558
0.4125	2.45723	5.00197	0.0625246
0.425	2.5196	5.1142	0.0639275
0.4375	2.58337	5.22925	0.0653656
0.45	2.64858	5.34716	0.0668395
0.4625	2.71527	5.46803	0.0683504
0.475	2.78346	5.59192	0.069899
0.4875	2.8532	5.7189	0.0714863
0.5	2.92453	5.84906	0.0731133
0.5125	2.99749	5.98248	0.074781

0.525	3.07211	6.11923	0.0764903
0.5375	3.14845	6.2594	0.0782425
0.55	3.22653	6.40307	0.0800384
0.5625	3.30642	6.55033	0.0818792
0.575	3.38814	6.70128	0.083766
0.5875	3.47175	6.856	0.0857
0.6	3.55729	7.01459	0.0876823
0.6125	3.64482	7.17714	0.0897142
0.625	3.73438	7.34375	0.0917969
0.6375	3.82602	7.51453	0.0939317
0.65	3.91979	7.68959	0.0961198
0.6625	4.01576	7.86901	0.0983627
0.675	4.11396	8.05293	0.100662
0.6875	4.21447	8.24144	0.103018
0.7	4.31733	8.43466	0.105433
0.7125	4.42261	8.63271	0.107909
0.725	4.53036	8.83572	0.110446
0.7375	4.64065	9.0438	0.113047
0.75	4.75354	9.25708	0.115714
0.7625	4.8691	9.4757	0.118446
0.775	4.98739	9.69978	0.121247
0.7875	5.10848	9.92946	0.124118
0.8	5.23244	10.1649	0.127061
0.8125	5.35935	10.4062	0.130077
0.825	5.48927	10.6535	0.133169
0.8375	5.62228	10.9071	0.136338
0.85	5.75846	11.1669	0.139587

0.8625	5.89789	11.4333	0.142916	
0.875	6.04065	11.7063	0.146329	
0.8875	6.18682	11.9861	0.149827	
0.9	6.3365	12.273	0.153412	
0.9125	6.48975	12.567	0.157088	
0.925	6.64668	12.8684	0.160855	
0.9375	6.80738	13.1773	0.164716	
0.95	6.97194	13.4939	0.168674	
0.9625	7.14046	13.8184	0.17273	
0.975	7.31303	14.1511	0.176888	
0.9875	7.48976	14.492	0.18115	
1.	7.67076	14.8415	0.185519	$\square$

"He is great whose faults can be numbered." - Hebrew proverb