Math 228
1-23-1994

Solutions to Problem Set 2.

[page 22]

The problems in this assignment involve the use of an integrating factor to solve the **first-order linear equation**

$$y' + P(x)y = Q(x).$$

The integrating factor is

$$I = \exp \int P(x)\, dx.$$

Compute I using this formula and obtain the equation

$$y \cdot I = \int IQ\, dx.$$

To finish, compute the integral on the right side and solve for y. The "I", "P", and "Q" in the solutions below correspond to the expressions in this discussion.

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3. Find the general solution of the differential equation $y' + y = xe^{-x} + 1$.

The integrating factor is

$$I = \exp \int dx = e^x.$$

Then

$$ye^x = \int (x + e^x)\, dx = \frac{1}{2}x^2 + e^x + C.$$

Therefore, the solution is

$$y = \frac{1}{2}x^2 e^{-x} + 1 + Ce^{-x}. \quad \square$$

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6. Find the general solution of the differential equation $xy' + 2y = \sin x$, $x > 0$.

Rewrite the equation as $y' + \dfrac{2}{x}y = \dfrac{\sin x}{x}$. The integrating factor is

$$I = \exp \int \frac{2}{x}\, dx = \exp 2\ln x = \exp \ln x^2 = x^2.$$

Hence,

$$yx^2 = \int x \sin x\, dx = -x\cos x + \sin x + C.$$

The solution is

$$y = -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x + \frac{C}{x^2}.$$

Here's the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & \sin x \\ & & \searrow & \\ - & 1 & & -\cos x \\ & & \searrow & \\ + & 0 & \rightarrow & -\sin x \end{array}$$

$$\int x \sin x \, dx = -x \cos x + \sin x + C. \quad \square$$

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9. Find the solution of the initial value problem $y' - y = 2xe^{2x}$, $y(0) = 1$.

The integrating factor is

$$I = \exp \int (-1) \, dx = e^{-x}.$$

Therefore,

$$ye^{-x} = \int 2xe^x \, dx = 2xe^x - 2e^x + C.$$

The general solution is

$$y = 2xe^{2x} - 2e^{2x} + Ce^x.$$

Here's the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & e^x \\ & & \searrow & \\ - & 1 & & e^x \\ & & \searrow & \\ + & 0 & \rightarrow & e^x \end{array}$$

$$\int xe^x \, dx = xe^x - e^x + C.$$

Plug in the initial condition and solve for the arbitrary constant:

$$1 = y(0) = 0 - 2 + C \quad \text{so} \quad C = 3.$$

The solution is

$$y = 2xe^{2x} - 2e^{2x} + 3e^x. \quad \square$$

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14. Find the solution of the initial value problem $xy' + 2y = \sin x$, $x > 0$, $y\left(\dfrac{\pi}{2}\right) = 1$.

From Problem 6,

$$y = -\frac{1}{x} \cos x + \frac{1}{x^2} \sin x + \frac{C}{x^2}.$$

Plug in the initial condition and solve for the arbitrary constant:

$$1 = y\left(\frac{\pi}{2}\right) = 0 + \frac{4}{\pi^2} + C \cdot \frac{4}{\pi^2} \quad \text{so} \quad C = \frac{\pi^2}{4} - 1.$$

The solution is

$$y = -\frac{1}{x} \cos x + \frac{1}{x^2} \sin x + \left(\frac{\pi^2}{4} - 1\right) \cdot \frac{1}{x^2}. \quad \square$$

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21. Find the solution of

$$\frac{dy}{dx} = \frac{1}{e^y - x}, \quad y(1) = 0.$$

Invert both sides of the equation:

$$\frac{dx}{dy} = e^y - x, \quad \text{or} \quad \frac{dx}{dy} + x = e^y.$$

The equation is linear in x. The integrating factor is

$$I = \exp \int dy = e^y.$$

Hence,

$$xe^y = \int e^{2y}\, dy = \frac{1}{2}e^{2y} + C.$$

The general solution is

$$x = \frac{1}{2}e^y + Ce^{-y}.$$

Plug in the initial condition and solve for the arbitrary constant:

$$1 = x = \frac{1}{2} + C \quad \text{so} \quad C = \frac{1}{2}.$$

The solution is

$$x = \frac{1}{2}e^y + \frac{1}{2}e^{-y}. \quad \square$$

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22. (a) Show that $\phi(x) = e^{2x}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(x)$ is also a solution of this equation for any value of the constant c.

If $y = c\phi(x) = ce^{2x}$, then

$$y' = 2ce^{2x} \quad \text{so} \quad y' - 2y = 2ce^{2x} - 2ce^{2x} = 0.$$

Hence, $y = c\phi(x) = ce^{2x}$ is a solution to the equation. ($\phi(x) = e^{2x}$ is the case $c = 1$.) $\square$

(b) Show that $\phi(x) = \dfrac{1}{x}$ is a solution of $y' + y^2 = 0$ for $x > 0$, but that $y = c\phi(x)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part (b) is nonlinear, while that of part (a) is linear.

If $y = \phi(x) = \dfrac{1}{x}$, then

$$y' = -\frac{1}{x^2} \quad \text{so} \quad y' + y^2 = -\frac{1}{x^2} + \frac{1}{x^2} = 0.$$

Thus, $y = \phi(x) = \dfrac{1}{x}$ is a solution to the equation.

However, if $y = c\phi(x) = \dfrac{c}{x}$, then

$$y' = -\frac{c}{x^2} \quad \text{so} \quad y' + y^2 = -\frac{c}{x^2} + \frac{c^2}{x^2} = \frac{c^2 - c}{x^2}.$$

$\dfrac{c^2 - c}{x^2}$ is identically 0 if and only if $c = 0$ or $c = 1$. Only in these cases is $y = c\phi(x) = \dfrac{c}{x}$ a solution. $c = 0$ gives $y = 0$, while $c = 1$ gives $y = \dfrac{1}{x}$. $\square$

"The ultimate test for us of what a truth means is the conduct it dictates or inspires." - William James