Math 228
4-12-1994

Solutions to Problem Set 30.

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2. Solve the differential equation $y'' - xy' - y = 0$ by means of a power series about the point $x_0 = 0$. Find the recurrence relation; also, find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} (n+1) a_n x^n = 0.$$

In the second sum, substitute $n = k - 2$. When $n = 0$, $k = 2$, and $n + 1 = k - 1$. Then

$$\sum_{n=0}^{\infty} (n+1) a_n x^n = \sum_{k=2}^{\infty} (k-1) a_{k-2} x^{k-2} = \sum_{n=2}^{\infty} (n-1) a_{n-2} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n. Now my equation is

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} (n-1) a_{n-2} x^{n-2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n \geq 2$, it implies that

$$n(n-1) a_n - (n-1) a_{n-2} = 0, \quad \text{or} \quad a_n = \frac{1}{n} a_{n-2}.$$

I can use this recursion formula to express a_n in terms of a_0 and a_1. There are two cases, depending on whether n is even or odd.

If I plug in even numbers for n, I get

$$\begin{aligned} a_2 &= \frac{1}{2} a_0, \\ a_4 &= \frac{1}{4} a_2, \\ &\vdots \\ a_{2n} &= \frac{1}{2n} a_{2n-2}. \end{aligned}$$

Multiply these equations together — that is multiply the left sides, multiply the right sides, and set the two products equal:

$$a_2 a_4 \cdots a_{2n} = \frac{1}{2} \cdot \frac{1}{4} \cdots \frac{1}{2n} \cdot a_0 a_2 \cdots a_{2n-2}.$$

Cancel the terms a_2, a_4, ..., a_{2n-2} from both sides:

$$a_{2n} = \frac{1}{2} \cdot \frac{1}{4} \cdots \frac{1}{2n} \cdot a_0.$$

The same trick works with the odd terms. If I plug in odd numbers for n, I get

$$\begin{aligned} a_3 &= \frac{1}{3} a_1, \\ a_5 &= \frac{1}{5} a_3, \\ &\vdots \\ a_{2n+1} &= \frac{1}{2n+1} a_{2n-1}. \end{aligned}$$

Multiply the left sides, multiply the right sides, and set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \frac{1}{3} \cdot \frac{1}{5} \cdots \frac{1}{2n+1} \cdot a_1 a_3 \cdots a_{2n-1}.$$

Cancel the terms a_3, a_5, ..., a_{2n-1} from both sides:

$$a_{2n+1} = \frac{1}{3} \cdot \frac{1}{5} \cdots \frac{1}{2n+1} \cdot a_1.$$

The solution is

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{4} \cdots \frac{1}{2n} x^{2n}\right) + a_1 \left(x + \sum_{n=1}^{\infty} \frac{1}{3} \cdot \frac{1}{5} \cdots \frac{1}{2n+1} x^{2n+1}\right). \quad \square$$

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5. Solve the differential equation $(1-x)y'' + y = 0$ by means of a power series about the point $x_0 = 0$. Find the recurrence relation; also, find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second sum, substitute $k - 2 = n - 1$, so $n = k - 1$. When $n = 0$, $k = 1$, so

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} = \sum_{k=1}^{\infty} (k-1)(k-2) a_{k-1} x^{k-2} = \sum_{n=1}^{\infty} (n-1)(n-2) a_{n-1} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
In the third sum, substitute $n = k - 2$. When $n = 0$, $k = 2$, so

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=2}^{\infty} a_{k-2} x^{k-2} = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

To get the second equality, I replaced the dummy variable k with n.
Now my equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty}(n-1)(n-2)a_{n-1}x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n \geq 2$, it implies that

$$n(n-1)a_n - (n-1)(n-2)a_{n-1} + a_{n-2} = 0, \quad \text{or} \quad a_n = \frac{-a_{n-2} + (n-1)(n-2)a_{n-1}}{n(n-1)}.$$

I can use this recursion formula to express a_n in terms of a_0 and a_1. It's easy to use *Mathematica* to compute as many terms as desired. I'll denote the n-th coefficient by `a5[n]`. Define the first two terms:

```
a5[0] = a50; a5[1] = a51;
```

Next, enter the recursion formula for the n-th term:

```
a5[n_] := a5[n] = Expand[
    (-a5[n-2] + (n - 1)(n - 2)a5[n-1])/(n(n - 1))]
```

Here is the tenth coefficient:

```
a5[10]

     -1463 a50   1319 a51
     --------- - --------
      518400      181440
```

The solution is

$$y = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^6 - \cdots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 - \frac{1}{40}x^6 - \cdots\right). \square$$

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9. Solve the differential equation $(1 + x^2)y'' - 4xy' + 6y = 0$ by means of a power series about the point $x_0 = 0$. Find the recurrence relation; also, find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - 4\sum_{n=0}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0.$$

In the last three sums, substitute $n = k - 2$, so $n - 1 = k - 3$, and when $n = 0$, $k = 2$. Here are the three sums:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{k=2}^{\infty}(k-2)(k-3)a_{k-2}x^{k-2} = \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2},$$

$$4\sum_{n=0}^{\infty} na_n x^n = 4\sum_{k=2}^{\infty}(k-2)a_{k-2}x^{k-2} = 4\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2},$$

$$6\sum_{n=0}^{\infty} a_n x^n = 6\sum_{k=2}^{\infty} a_{k-2}x^{k-2} = 6\sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

(In all three, I substituted n for the dummy variable k in the second step.)

These three sums add to give

$$\sum_{n=2}^{\infty}\left((n-2)(n-3) - 4(n-2) + 6\right) a_{n-2}x^{n-2} = \sum_{n=2}^{\infty}(n-4)(n-5)a_{n-2}x^{n-2}.$$

So the differential equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty}(n-4)(n-5)a_{n-2}x^{n-2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n \ge 2$, it implies that

$$n(n-1)a_n + (n-4)(n-5)a_{n-2} = 0, \quad \text{or} \quad a_n = -\frac{(n-4)(n-5)}{n(n-1)}a_{n-2}.$$

I can use this recursion formula to express a_n in terms of a_0 and a_1. There are two cases, depending on whether n is even or odd.

If I plug in even numbers for n, I get

$$\begin{aligned} a_2 &= -3a_0, \\ a_4 &= 0, \\ a_6 &= -\frac{2}{30}a_4 = 0, \\ &\vdots \\ a_{2n} &= 0. \end{aligned}$$

If I plug in odd numbers for n, I get

$$\begin{aligned} a_3 &= -\frac{1}{3}a_1, \\ a_5 &= 0, \\ a_7 &= -\frac{6}{42}a_5 = 0, \\ &\vdots \\ a_{2n+1} &= 0. \end{aligned}$$

The solution is

$$y = a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3 = a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right). \square$$

14. Solve the differential equation $2y'' + (x+1)y' + 3y = 0$ by means of a power series about the point $x_0 = 2$. Find the recurrence relation; also, find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

Substitute $u = x - 2$. When $x = 2$, $u = 0$, and $x + 1 = u + 3$. Moreover,

$$\frac{dy}{dx} = \frac{dy}{du} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2y}{du^2}.$$

The differential equation becomes

$$2y'' + (u+3)y' + 3y = 0.$$

(I'm abusing notation with y'' and y', but the previous observation shows there's no harm.)
Set

$$y = \sum_{n=0}^{\infty} a_n u^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n u^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n u^{n-2}.$$

Substitute:

$$2\sum_{n=0}^{\infty} n(n-1)a_n u^{n-2} + \sum_{n=0}^{\infty} n a_n u^n + 3\sum_{n=0}^{\infty} n a_n u^{n-1} + 3\sum_{n=0}^{\infty} a_n u^n = 0.$$

In the second and fourth sums, substitute $n = k - 2$. When $n = 0$, $k = 2$, so the sums become

$$\sum_{n=0}^{\infty} n a_n u^n = \sum_{k=2}^{\infty} (k-2)a_{k-2}u^{k-2} = \sum_{n=2}^{\infty} (n-2)a_{n-2}u^{n-2},$$

$$3\sum_{n=0}^{\infty} a_n u^n = 3\sum_{k=2}^{\infty} a_{k-2}u^{k-2} = 3\sum_{n=2}^{\infty} a_{n-2}u^{n-2}.$$

In the third sum, substitute $k - 2 = n - 1$, so $n = k - 1$. When $n = 0$, $k = 1$, so the sum becomes

$$3\sum_{n=0}^{\infty} n a_n u^{n-1} = 3\sum_{k=1}^{\infty} (k-1)a_{k-1}u^{k-2} = 3\sum_{n=1}^{\infty} (n-1)a_{n-1}u^{n-2}.$$

In each case, to get the second equality, I replaced the dummy variable k with n.
Now my equation is

$$2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-2)a_{n-2}u^{n-2} + 3\sum_{n=1}^{\infty} (n-1)a_{n-1}u^{n-2} + 3\sum_{n=2}^{\infty} a_{n-2}u^{n-2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n \geq 2$, it implies that

$$2n(n-1)a_n + (n-2)a_{n-2} + 3(n-1)a_{n-1} + 3a_{n-2} = 0.$$

Therefore, for $n \geq 2$,

$$a_n = -\frac{(n+1)a_{n-2} + 3(n-1)a_{n-1}}{2n(n-1)}.$$

I can use this recursion formula to express a_n in terms of a_0 and a_1.
Writing down the series in u, then substituting $u = x - 2$, I find that the solution is

$$y = a_0\left(1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 - \cdots\right) +$$

$$a_1\left((x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 - \cdots\right). \square$$